MetaGrad

Multiple Learning Rates in Online Learning

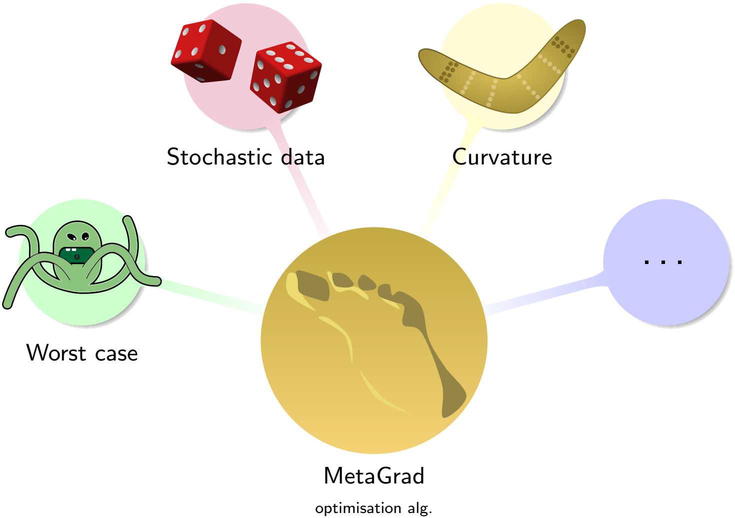
http://bitbucket.org/wmkoolen/metagrad

Tim van Erven
Universiteit Leiden

Wouter M. Koolen
CWI
Centrum Wiskunde & Informatica

NIPS, Barcelona
Tuesday 6th December, 2016

In a Nutshell



Optimisation Pervasive in Machine Learning

$$\min_{\boldsymbol{w}} \sum_{t=1}^{T} f_t(\boldsymbol{w})$$

Optimisation Pervasive in Machine Learning

$$\min_{w} \sum_{t=1}^{T} f_t(w)$$

Batch Training
(classification)

Optimisation Pervasive in Machine Learning

$$\min_{\boldsymbol{w}} \sum_{t=1}^{T} f_t(\boldsymbol{w})$$

Batch Training
(classification)

Time Series
(investment)

Optimisation Pervasive in Machine Learning

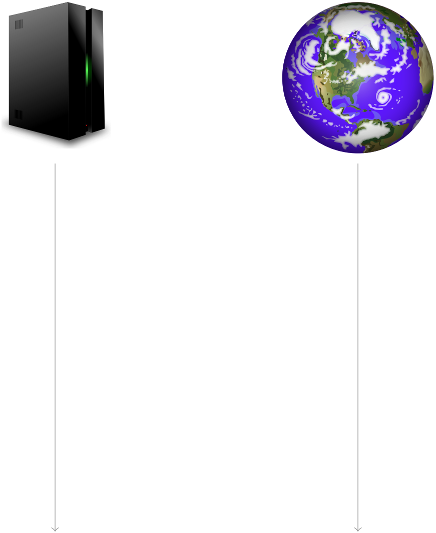
$$\min_{w} \sum_{t=1}^{T} f_t(w)$$

Batch Training (classification)

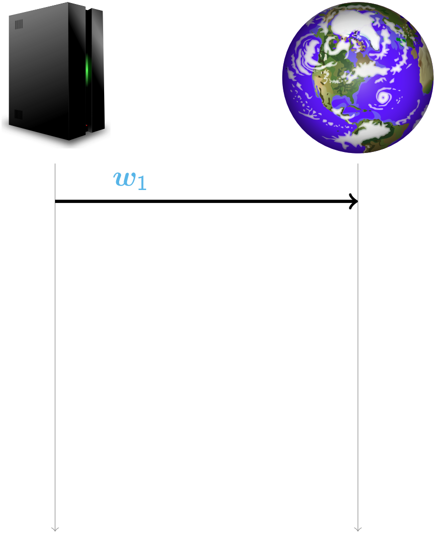
Time Series (investment)

Big Data

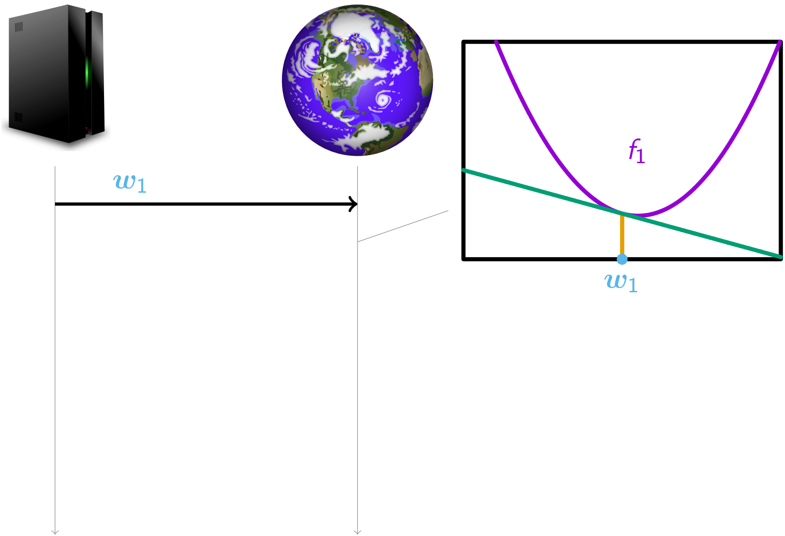
Online Convex Optimisation



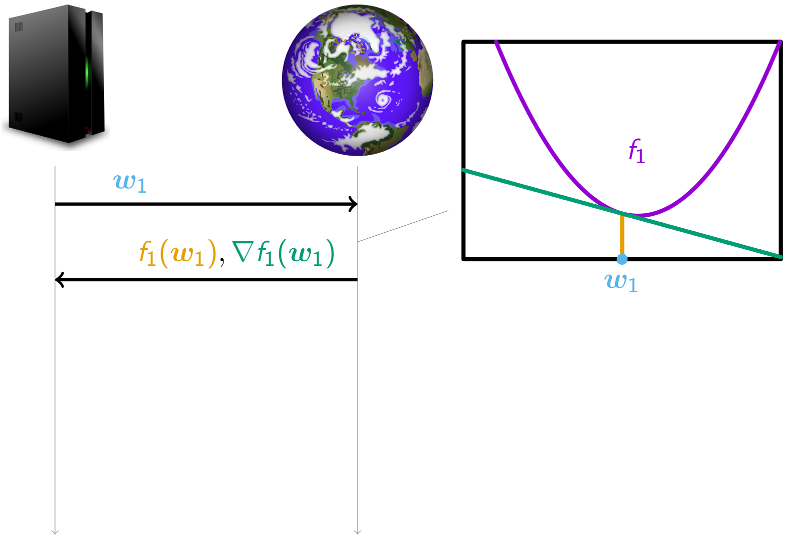
Online Convex Optimisation



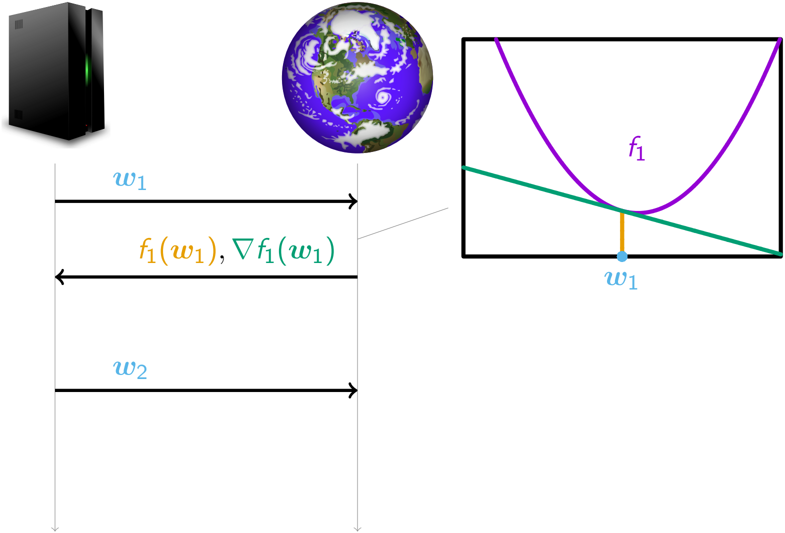
Online Convex Optimisation



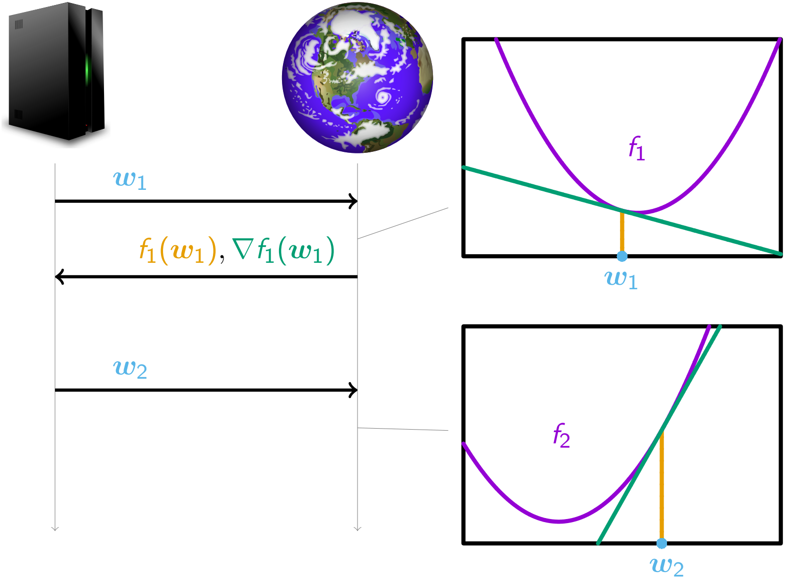
Online Convex Optimisation



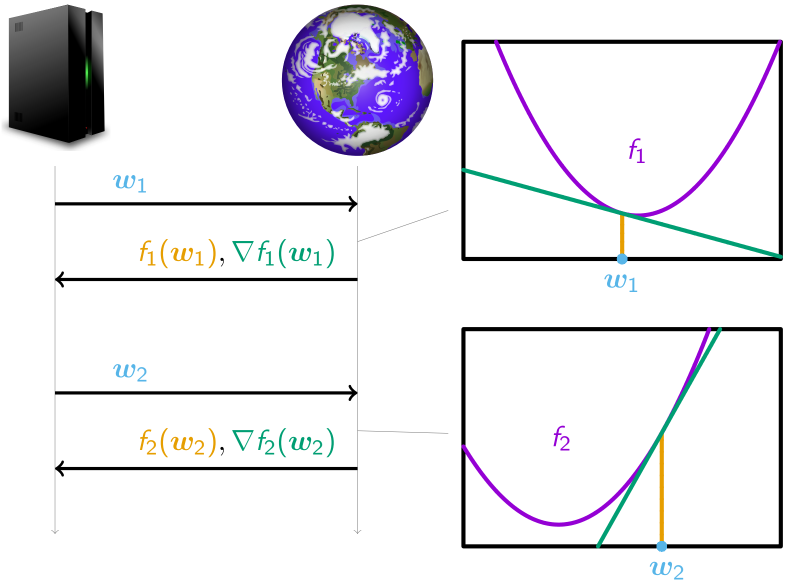
Online Convex Optimisation



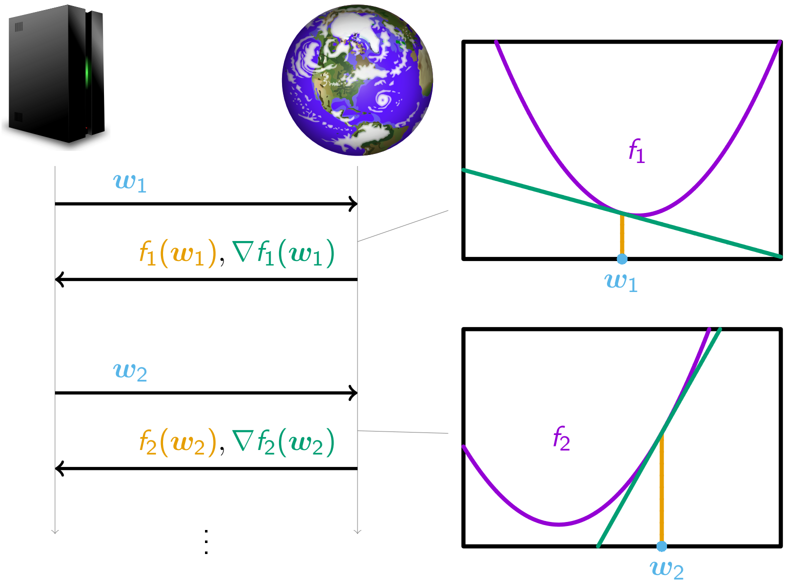
Online Convex Optimisation



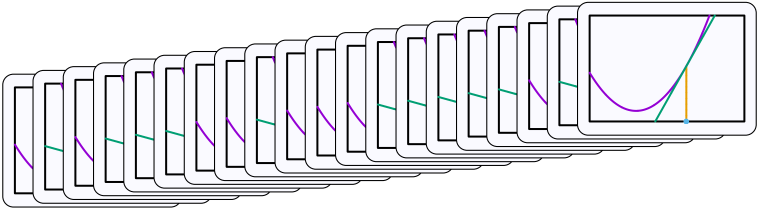
Online Convex Optimisation



Online Convex Optimisation



Objective



Definition (Regret)

$$R_T = \underbrace{\sum_{t=1}^{T} f_t(\boldsymbol{w}_t)}_{\text{Online loss}} - \underbrace{\min_{\boldsymbol{u}} \sum_{t=1}^{T} f_t(\boldsymbol{u})}_{\text{Optimal loss}}$$

Online Gradient Descent [Zinkevich, 2003]

$$\boldsymbol{w}_{t+1} = \boldsymbol{w}_t - \eta_t \nabla f_t(\boldsymbol{w}_t)$$

Online Gradient Descent [Zinkevich, 2003]

$$\boldsymbol{w}_{t+1} = \boldsymbol{w}_t - \eta_t \nabla f_t(\boldsymbol{w}_t)$$

Worst-case regret guarantee:

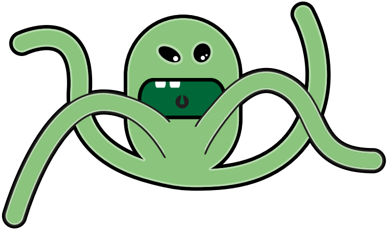
$$R_T = O\left(\sqrt{T}\right)$$

Online Gradient Descent [Zinkevich, 2003]

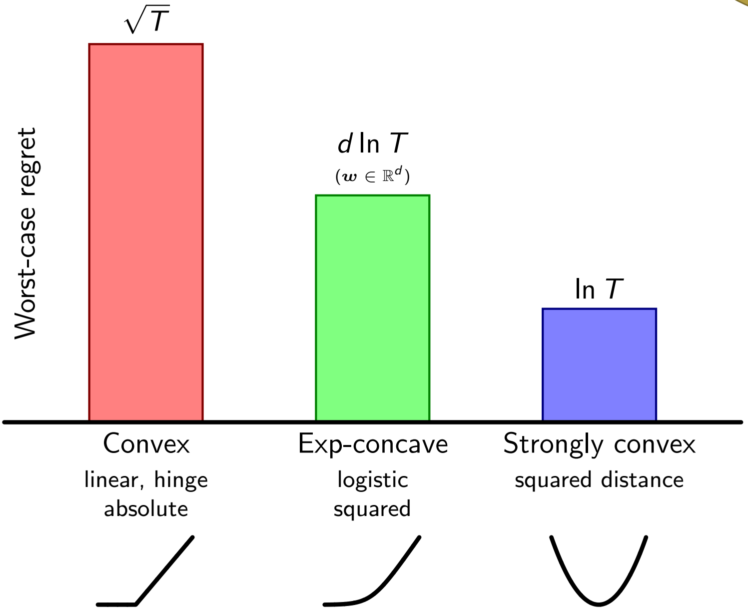
$$\boldsymbol{w}_{t+1} = \boldsymbol{w}_t - \eta_t \nabla f_t(\boldsymbol{w}_t)$$

Worst-case regret guarantee:

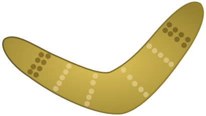
$$R_T = O\left(\sqrt{T}\right)$$

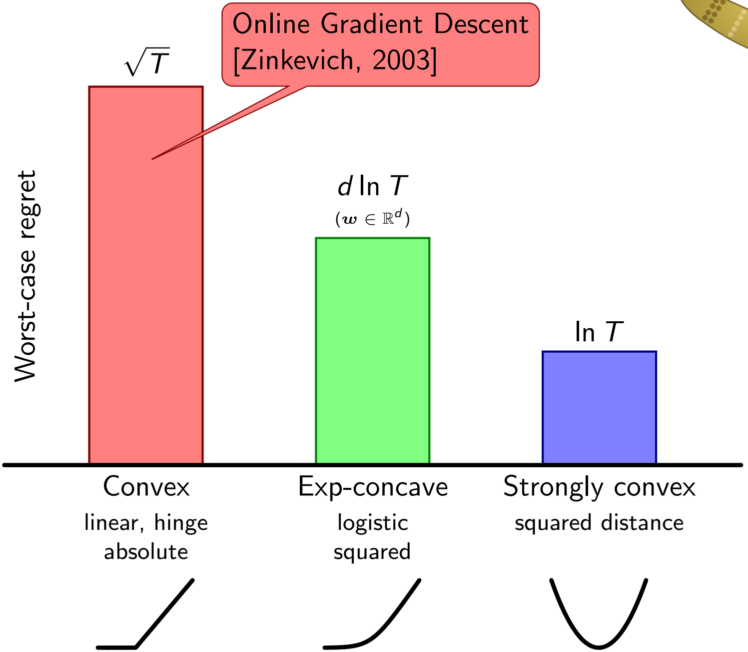


Loss Taxonomy ∼ Curvature

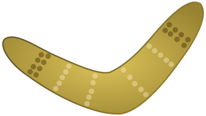


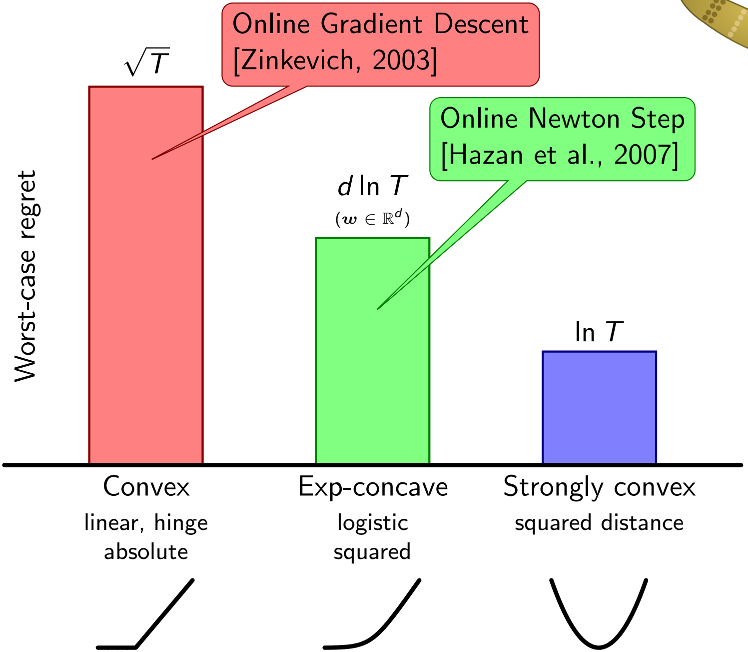
Loss Taxonomy $\sim$ Curvature



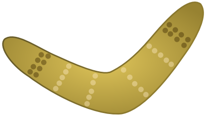


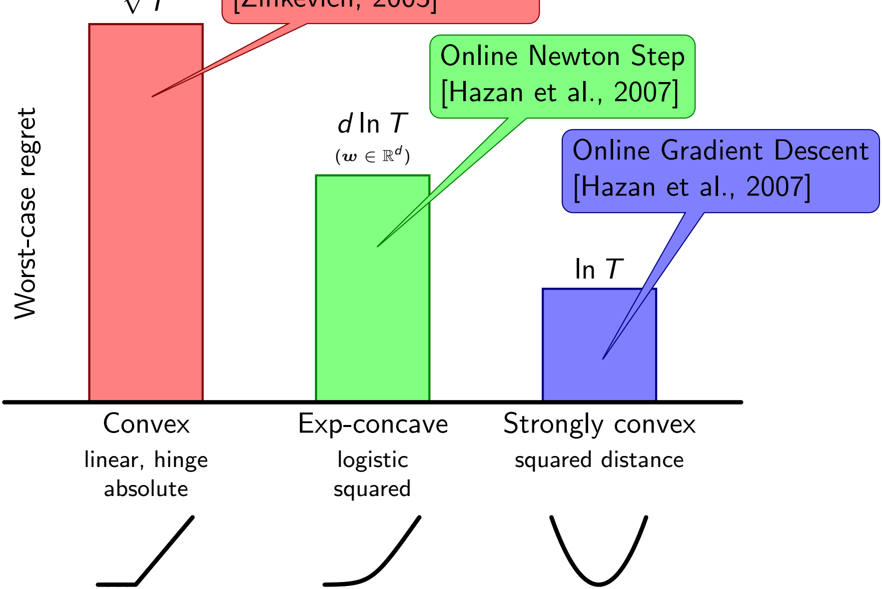
Loss Taxonomy ∼ Curvature





Loss Taxonomy $\sim$ Curvature





Big Questions

Can we make **adaptive** methods for **online convex optimisation** that are

- **worst-case safe**
- exploit **curvature** automatically
- computationally **efficient**

Big Questions

Can we make **adaptive** methods for **online convex optimisation** that are

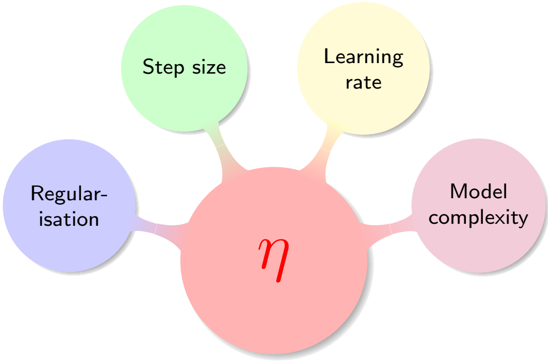
- **worst-case safe**
- exploit **curvature** automatically
- computationally **efficient**

And can we adapt to other important regimes?

- **Mixed** or **in-between** cases?
- **Stochastic** data? Bandits [Seldin and Slivkins, 2014]
- Absence of **curvature**? Experts [Koolen and Van Erven, 2015]

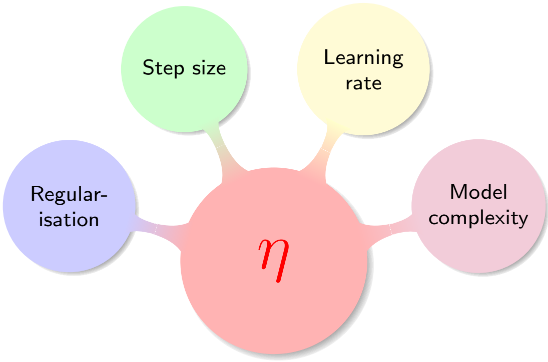
Main Idea

For every optimisation algorithm tuning is **crucial**.



Main Idea

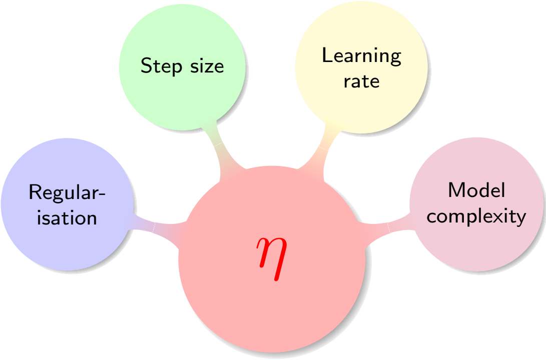
For every optimisation algorithm tuning is **crucial**.



So let's **learn** optimal tuning from **data**.

Main Idea

For every optimisation algorithm tuning is **crucial**.

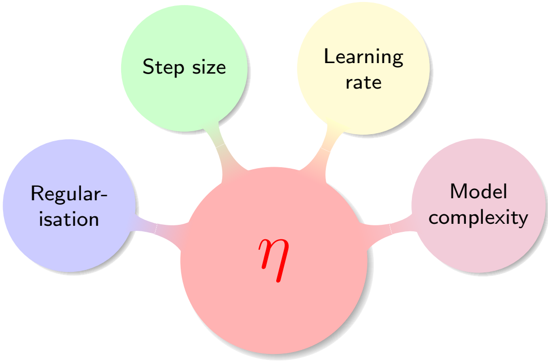


So let's **learn** optimal tuning from **data**.

Key obstacle: avoid learning η at **slow rate** itself.

Main Idea

For every optimisation algorithm tuning is **crucial**.

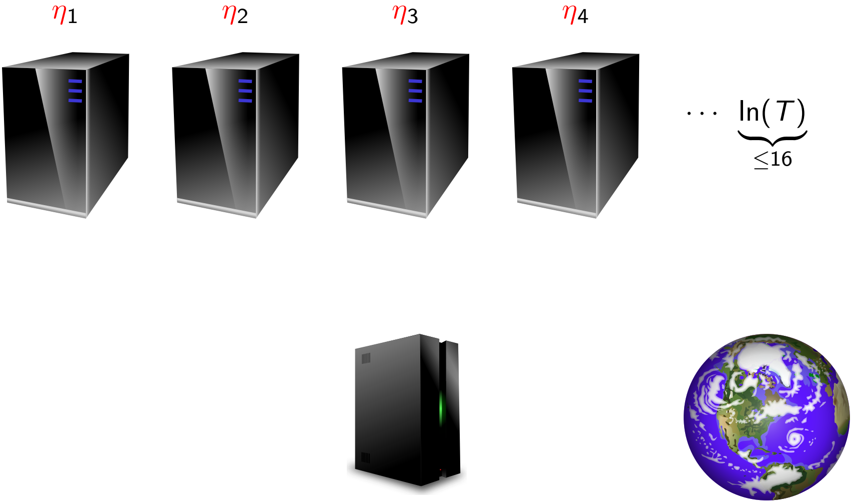


So let's **learn** optimal tuning from **data**.

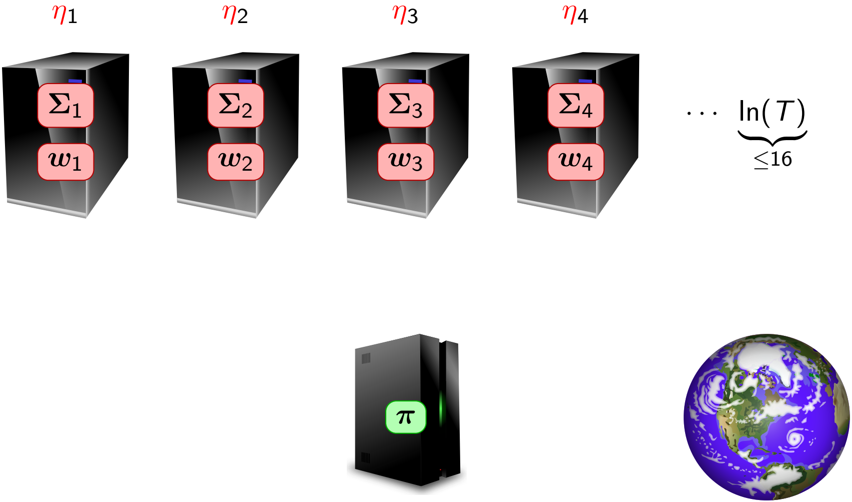
Key obstacle: avoid learning η at **slow rate** itself.

Breakthrough: Multiple Eta Gradient algorithm (MetaGrad)

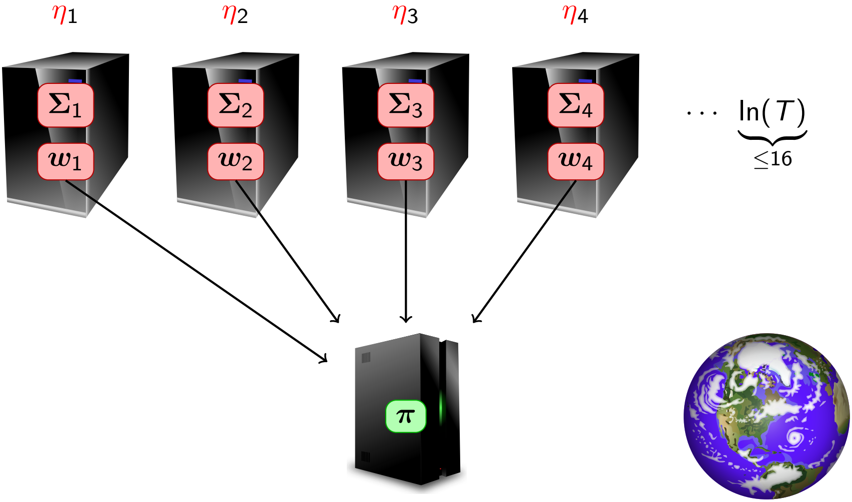
MetaGrad Algorithm



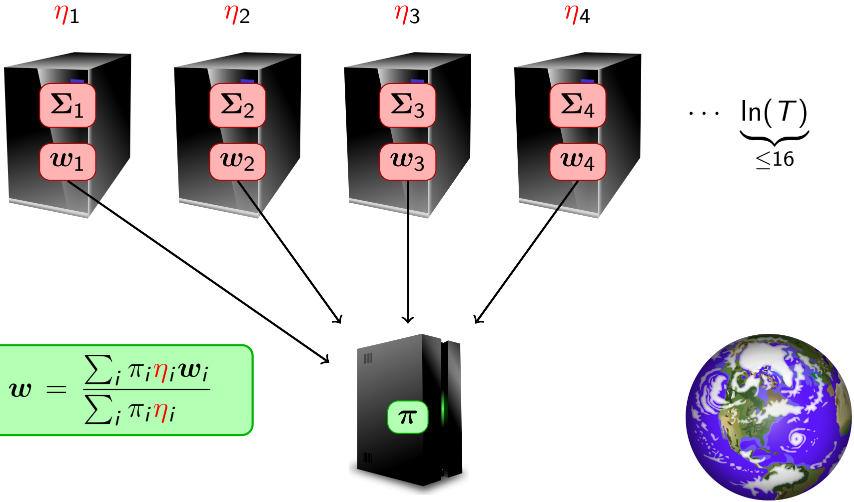
MetaGrad Algorithm



MetaGrad Algorithm

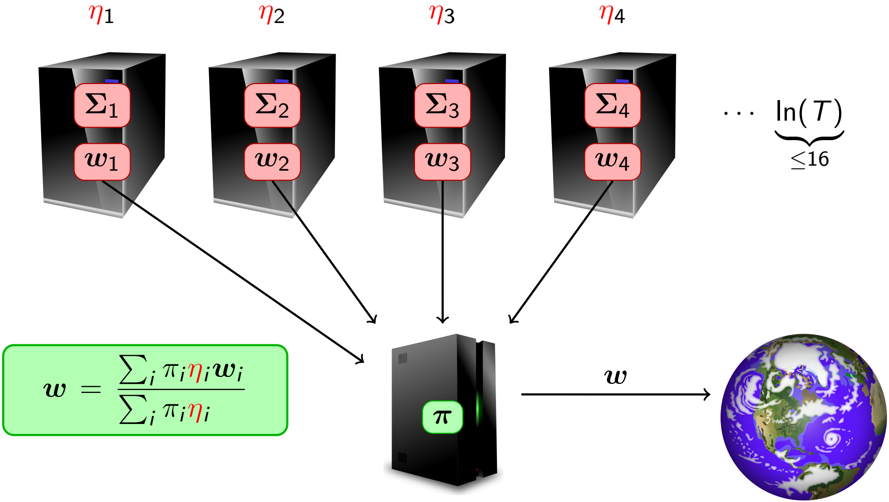


MetaGrad Algorithm



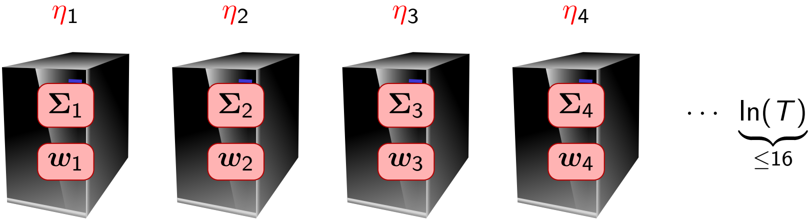
$$\boldsymbol{w} = \frac{\sum_i \pi_i \eta_i \boldsymbol{w}_i}{\sum_i \pi_i \eta_i}$$

MetaGrad Algorithm

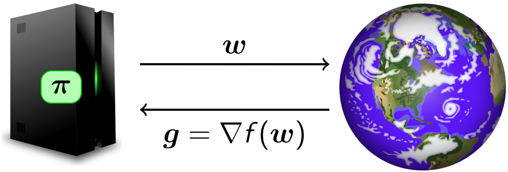


$$\boldsymbol{w} = \frac{\sum_i \pi_i \eta_i \boldsymbol{w}_i}{\sum_i \pi_i \eta_i}$$

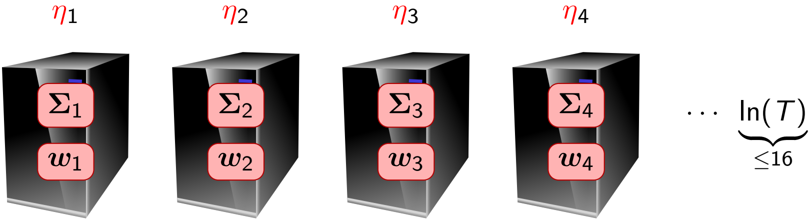
MetaGrad Algorithm



$$\boldsymbol{w} = \frac{\sum_i \pi_i \eta_i \boldsymbol{w}_i}{\sum_i \pi_i \eta_i}$$



MetaGrad Algorithm

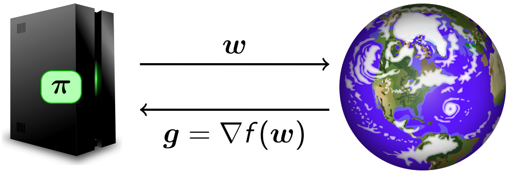


$$w = \frac{\sum_i \pi_i \eta_i w_i}{\sum_i \pi_i \eta_i}$$

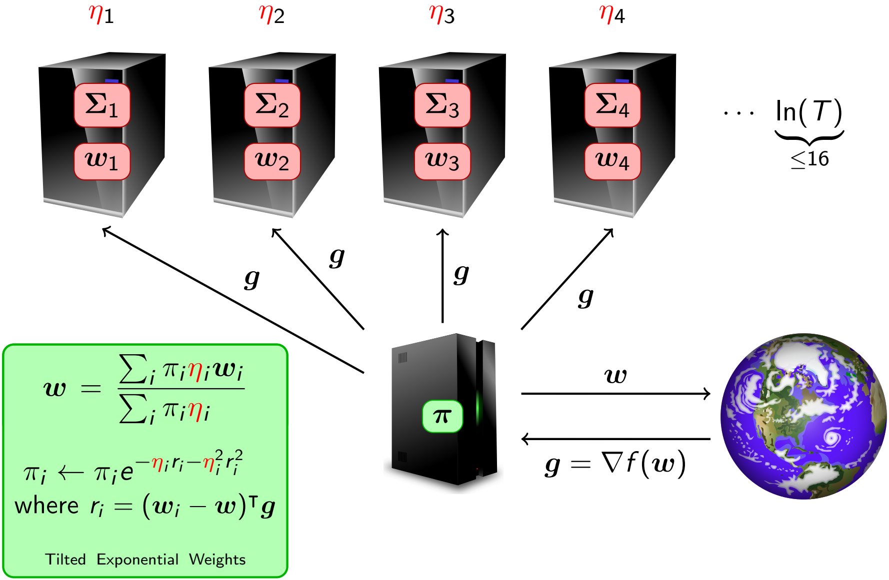
$$\pi_i \leftarrow \pi_i e^{-\eta_i r_i - \eta_i^2 r_i^2}$$

where $r_i = (w_i - w)^\intercal g$

Tilted Exponential Weights



MetaGrad Algorithm

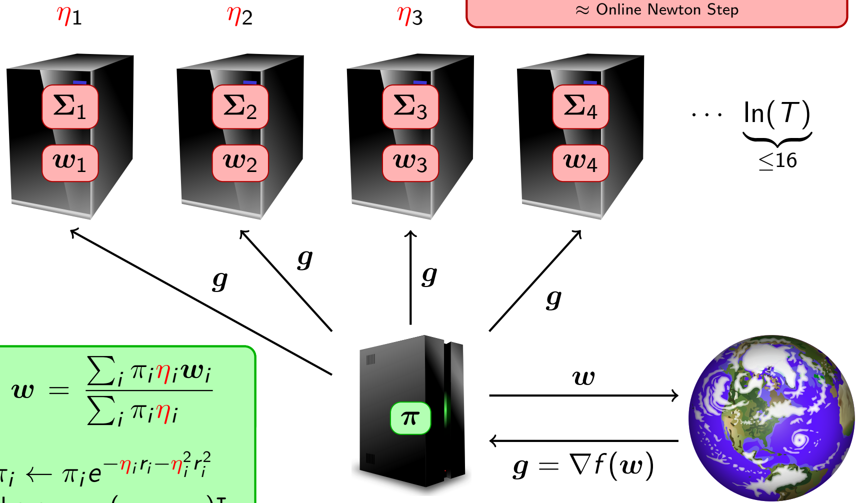


MetaGrad Algorithm

$$\boldsymbol{\Sigma}_i \leftarrow (\boldsymbol{\Sigma}_i^{-1} + 2\eta_i^2 \boldsymbol{g}\boldsymbol{g}^\intercal)^{-1}$$

$$\boldsymbol{w}_i \leftarrow \boldsymbol{w}_i - \eta_i \boldsymbol{\Sigma}_i \boldsymbol{g}\,(1 + 2\eta_i r_i)$$

≈ Online Newton Step



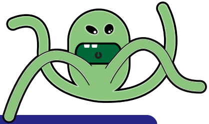
$$\boldsymbol{w} = \frac{\sum_i \pi_i \eta_i \boldsymbol{w}_i}{\sum_i \pi_i \eta_i}$$

$$\pi_i \leftarrow \pi_i e^{-\eta_i r_i - \eta_i^2 r_i^2}$$

where $r_i = (\boldsymbol{w}_i - \boldsymbol{w})^\intercal \boldsymbol{g}$

Tilted Exponential Weights

Second-order Regret Bound



Theorem

The regret of MetaGrad is bounded by

$$R_T \;=\; O\left(\min\left\{\sqrt{T}, \sqrt{\color{red}{V_T}\, d \ln T}\right\}\right),$$

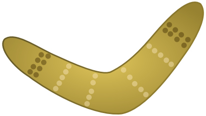
where

$$\color{red}{V_T} \;=\; \sum_{t=1}^{T} \left((\boldsymbol{w}_t - \boldsymbol{u}^*)^\intercal \nabla f_t(\boldsymbol{w}_t)\right)^2$$

measures **variance** *compared to the* **offline optimum** $\boldsymbol{u}^* = \arg\min_{\boldsymbol{u}} \sum_{t=1}^{T} f_t(\boldsymbol{u})$

Note: Optimal tuning depends on unknown optimum $\boldsymbol{u}^*$.

MetaGrad Adapts to Curvature



MetaGrad regret bound:

$$R_T = O\left(\sqrt{V_T \, d \ln T}\right)$$

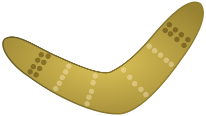
Corollary

For α-**exp-concave** *or* α-**strongly convex** *losses, MetaGrad ensures*

$$R_T = O(d \ln T)$$

without knowing α.

MetaGrad Adapts to Curvature



MetaGrad regret bound:

$$R_T \;=\; O\left(\sqrt{V_T\, d \ln T}\right)$$

Corollary

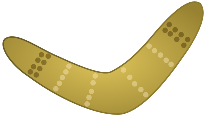
*For α-**exp-concave** or α-**strongly convex** losses, MetaGrad ensures*

$$R_T \;=\; O\left(d \ln T\right)$$

without knowing α.

Same result for fixed $f_t = f$ (classical optimisation) even without curvature via **derivative condition**.

MetaGrad Adapts to Curvature



MetaGrad regret bound:

$$R_T \;=\; O\left(\sqrt{V_T\, d \ln T}\right)$$

Corollary

*For α-**exp-concave** or α-**strongly convex** losses, MetaGrad ensures*

$$R_T \;=\; O\left(d \ln T\right)$$

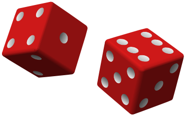
without knowing α.

Same result for fixed $f_t = f$ (classical optimisation) even without curvature via **derivative condition**.

Reason

Curvature implies $\Omega(V_T)$ cumulative slack between loss and its tangent lower bound.

MetaGrad Adapts to Stochastic Margin



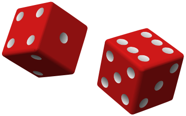
Consider i.i.d. losses $f_t \sim \mathbb{P}$ with **stochastic optimum**

$$\boldsymbol{u}^* = \arg\min_{\boldsymbol{u}} \mathbb{E}\, f(\boldsymbol{u})$$

Goal is small **pseudo-regret** compared to $\boldsymbol{u}^*$:

$$R_T^* = \sum_{t=1}^{T} f_t(\boldsymbol{w}_t) - \sum_{t=1}^{T} f_t(\boldsymbol{u}^*)$$

MetaGrad Adapts to Stochastic Margin



Consider i.i.d. losses $f_t \sim \mathbb{P}$ with **stochastic optimum**

$$\boldsymbol{u}^* = \arg\min_{\boldsymbol{u}} \mathbb{E} f(\boldsymbol{u})$$

Goal is small **pseudo-regret** compared to $\boldsymbol{u}^*$:

$$R_T^* = \sum_{t=1}^{T} f_t(\boldsymbol{w}_t) - \sum_{t=1}^{T} f_t(\boldsymbol{u}^*)$$

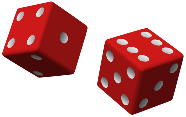
Corollary

For any β-**Bernstein** $\mathbb{P}$, *MetaGrad keeps the expected regret below*

$$\mathbb{E} R_T^* \leq O\left((d \ln T)^{\frac{1}{2-\beta}} \; T^{\frac{1-\beta}{2-\beta}}\right).$$

Fast rates without curvature: e.g. absolute loss, hinge loss, ...

MetaGrad Adapts to Stochastic Margin



Consider i.i.d. losses $f_t \sim \mathbb{P}$ with **stochastic optimum**

$$\boldsymbol{u}^* = \arg\min_{\boldsymbol{u}} \mathbb{E}\, f(\boldsymbol{u})$$

Goal is small **pseudo-regret** compared to $\boldsymbol{u}^*$:

$$R_T^* = \sum_{t=1}^{T} f_t(\boldsymbol{w}_t) - \sum_{t=1}^{T} f_t(\boldsymbol{u}^*)$$

Corollary

For any β-**Bernstein** $\mathbb{P}$, *MetaGrad keeps the expected regret below*

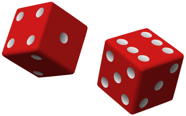
$$\mathbb{E}\, R_T^* \le O\left((d \ln T)^{\frac{1}{2-\beta}}\, T^{\frac{1-\beta}{2-\beta}}\right).$$

Fast rates without curvature: e.g. absolute loss, hinge loss, ...

Reason

Bernstein bounds $\mathbb{E}[V_T^*]$ above by $\mathbb{E}[R_T^*]$. "Solve" regret bound.

MetaGrad Adapts to Stochastic Margin



Consider i.i.d. losses $f_t \sim \mathbb{P}$ with **stochastic optimum**

$$\boldsymbol{u}^* = \arg\min_{\boldsymbol{u}} \mathbb{E} f(\boldsymbol{u})$$

Goal is small **pseudo-regret** compared to $\boldsymbol{u}^*$:

$$R_T^* = \sum_{t=1}^{T} f_t(\boldsymbol{w}_t) - \sum_{t=1}^{T} f_t(\boldsymbol{u}^*)$$

Corollary

*For any β-***Bernstein** $\mathbb{P}$*, MetaGrad keeps the ex... ...gret below*

$$\mathbb{E} R_T^* \le O\Big((d \ln T)^{\frac{1}{2-\beta}} \ldots$$

Joint work with P. Grünwald
Come see more at poster #76

Fast rates without curvature: e.g. abso... loss, hinge loss, ...

Reason

Bernstein bounds $\mathbb{E}[V_T^*]$ above by $\mathbb{E}[R_T^*]$. "Solve" regret bound.

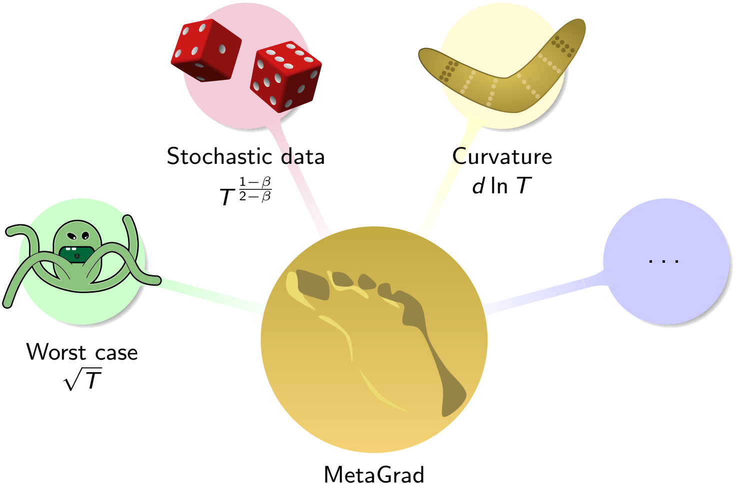
Conclusion

First contact with a new generation of **adaptive algorithms**.

Conclusion

First contact with a new generation of **adaptive algorithms**.

MetaGrad adapts to a wide range of environments:



Conclusion

First contact with a new generation of **adaptive algorithms**.

MetaGrad adapts to a wide range of [illegible]ments: