Buy low, sell high

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Centrum Wiskunde & Informatica
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Credits (Alphabetical)

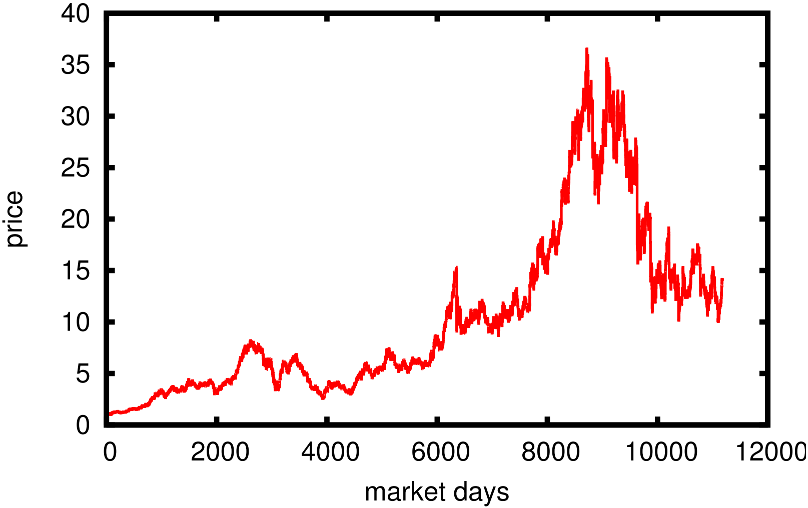
Part 1

- A. Philip Dawid
- Steven de Rooij
- Peter Grünwald
- Wouter M. Koolen
- Glenn Shafer
- Alexander Shen
- Nikolai Vereshchagin
- Vladimir Vovk

Part 2

- Wouter M. Koolen
- Vladimir Vovk

Example: Price of Kodak traded on NYSE '62 – '06



At a high level

Worst-case guarantees for finance

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Goals in this talk

1. Sell high
2. Buy low, sell high

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Goals in this talk

1. Sell high
2. Buy low, sell high

There is some overhead for not knowing the perfect trading time(s)

We *characterize* the achievable overheads
(using a surprisingly elegant formula)

We find a *canonical representation* of achievable guarantees.

Outline

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We consider trading in a single security (with initial price 1$/share)
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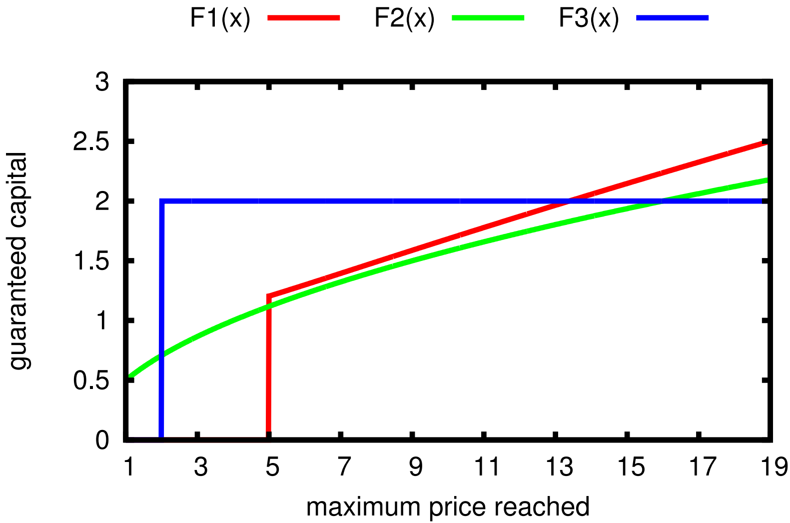
- Is guaranteeing F possible?
- Can more than F be guaranteed?
- Can we reverse engineer a strategy for F?

Example guarantees F



Protocol

Initial capital $K_0 := 1$
Initial price $\omega_0 := 1$

For day $t = 1, 2, \ldots$

1. **Investor** takes position $S_t \in \mathbb{R}$
2. **Market** reveals price $\omega_t \in [0, \infty)$
3. Capital becomes $K_t := K_{t-1} + S_t(\omega_t - \omega_{t-1})$

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- $S_t < 0$ is called short
- $S_t > 0$ is called long
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Bankrupt when capital $K_t < 0$ is negative.

No assumptions about price-generating process. Full information

Adjuster

A strategy prescribes position S_t based on the past prices $\omega_0, \ldots, \omega_{t-1}$.

Definition

A function $F : [1, \infty) \to [0, \infty)$ is called an adjuster if there is a strategy that guarantees

$$K_t \;\geq\; F\left(\max_{0 \leq s \leq t} \omega_s\right).$$

An adjuster F is admissible if it is not strictly dominated.

Threshold adjusters

Fix a price level $u \geq 1$. The threshold adjuster

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is witnessed by the threshold strategy S_u that

- takes position 1 until the price first exceeds level u.
- takes position 0 thereafter

The GUT of Adjusters

Consider a right-continuous and increasing *candidate guarantee* F.

Theorem (Characterisation)

F is an **adjuster** *iff*

$$\int_1^\infty \frac{F(y)}{y^2} \, \mathrm{d}y \;\leq\; 1.$$

Moreover, F is **admissible** *iff this holds with equality.*

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Theorem (Representation)

F is an **adjuster** *iff there is a probability measure P on $[1, \infty)$ such that*

$$F(y) \ \le \ \int F_u(y) \, \mathrm{d}P(u),$$

again with equality iff F is admissible.

Upper bound flavour

Fix an adjuster F, and consider the strategy that witnesses F. If the price $(\omega_t)_{t\geq 0}$ is a *martingale*, then so is the capital $(K_t)_{t\geq 0}$.

For each t, we must have

$$1 \;=\; K_0 \;=\; \mathbb{E}\left[K_t\right] \;\geq\; \mathbb{E}\left[F\left(\max_{0\leq s\leq t}\omega_s\right)\right]$$

For many martingales (e.g. Brownian motion), the random variable $\max_{0\leq s\leq t}\omega_s$ has density $\frac{1}{h^2}$ on $[1,\infty)$, so

$$\mathbb{E}\left[F\left(\max_{0\leq s\leq t}\omega_s\right)\right] \;=\; \int_1^\infty \frac{F(h)}{h^2}\,\mathrm{d}h$$

Lower bound flavour

Now take a differentiable F for which $\int_1^\infty F(y) y^{-2} \, \mathrm{d}y = 1$.

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Solve for weights $p_{\{1\}}$ and $p(u)$ in

$$F(y) \;=\; p_{\{1\}} F_1(y) + \int_1^\infty F_u(y) p(u) \, \mathrm{d}u \;=\; p_{\{1\}} + \int_1^y u p(u) \, \mathrm{d}u$$

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We get

$$p_{\{1\}} = F(1) \qquad \text{and} \qquad p(u) = \frac{F'(u)}{u}$$

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Now verify that weights are *probabilities*:

$$p_{\{1\}} + \int_1^\infty p(u)\,\mathrm{d}u \;=\; F(1) + \int_1^\infty \frac{F'(u)}{u}\,\mathrm{d}u \;=\; \int_1^\infty \frac{F(u)}{u^2}\,\mathrm{d}u \;=\; 1$$

What just happened

Sell high:

- We classified candidate guarantees using a simple formula
  - (≤ 1) Attainable adjuster
  - $(= 1)$ Admissible adjuster
  - (> 1) Not an adjuster

  We reverse engineered a strategy for each guarantee
  - Mixture of threshold (sell-at-level) strategies

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([a, b] is upcrossed when the price drops below a before it exceeds b)

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Definition

A price path $\omega_0, \ldots, \omega_t$ upcrosses interval $[a, b]$ if

$$\text{there are } 0 \leq t_a \leq t_b \leq t \quad \text{s.t.} \quad \omega_{t_a} \leq a \text{ and } \omega_{t_b} \geq b.$$

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A strategy prescribes position S_t based on the past prices $\omega_0, \dots, \omega_{t-1}$.

Definition

A function $G : (0, 1] \times [0, \infty) \to [0, \infty)$ is called an adjuster if there is a strategy that guarantees

$$K_t \;\ge\; G(a, b)$$

for each $[a, b]$ upcrossed by $\omega_0, \dots, \omega_t$.
An adjuster G is admissible if it is not strictly dominated.

Fallacy

- More of the same
- Fix price levels $\alpha < \beta$. The threshold adjuster

$$G_{\alpha,\beta}(a, b) \;=\; \frac{\beta}{\alpha}\mathbf{1}_{\{a \leq \alpha\}}\mathbf{1}_{\{b \geq \beta\}}$$

is witnessed by the threshold strategy $S_{\alpha,\beta}$ that

  - takes position 0 until the price drops below α
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- Optimal strategies allocate their 1\$ to threshold strategies according to some probability measure $P(\alpha, \beta)$, and hence achieve

$$G_P(a, b) \; = \; \int G_{\alpha,\beta}(a, b) \, \mathrm{d}P(\alpha, \beta).$$

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G_P is typically strictly dominated

Mixtures of thresholds are generally dominated

$$G(a,b) \;:=\; \frac{1}{2}G_{1,2}(a,b)+\frac{1}{2}G_{\frac{1}{2},1}(a,b) \;=\; \mathbf{1}_{\{a \le 1 \text{ and } b \ge 2\}}+\mathbf{1}_{\left\{a \le \frac{1}{2} \text{ and } b \ge 1\right\}}.$$

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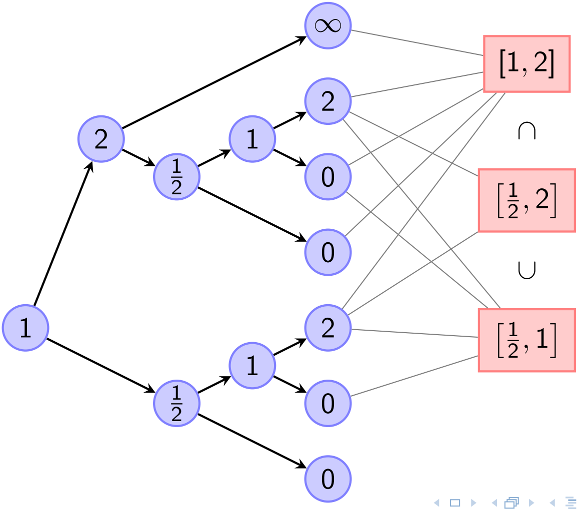

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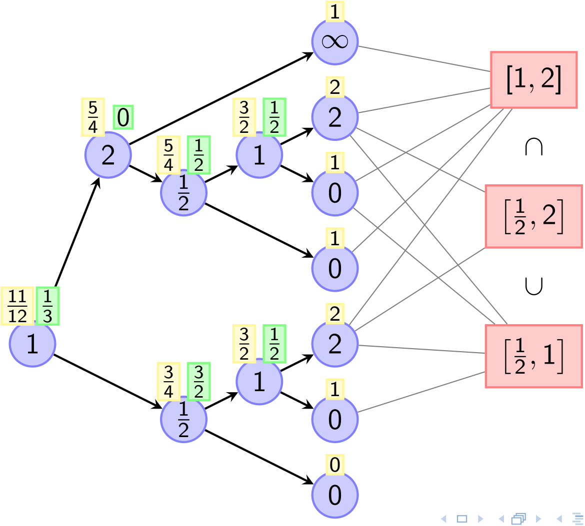

The GUT of Adjusters

Let G be left/right continuous and de/increasing.

Theorem (Characterisation)

G is an **adjuster** *iff*

$$\int_0^\infty 1 - \exp\left(\int_0^1 \frac{1}{a - \inf\{b \mid G(a,b) \geq h\}} \,\mathrm{d}a\right) \mathrm{d}h \;\leq\; 1.$$

Moreover, G is **admissible** *iff this holds with equality.*

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Moreover, G is **admissible** *iff this holds with equality.*

Theorem (Representation)

G is an **adjuster** *iff there are a probability measure Q on $[0, \infty)$ and a nested family $(I_h)_{h \geq 0}$ of north-west sets such that*

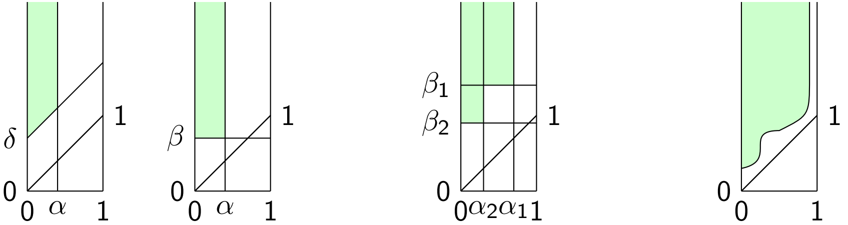
$$G(a,b) \;\leq\; \int G_{I_h}(a,b) \,\mathrm{d}Q(h),$$

with equality iff G is **admissible**.

Reverse engineering I

Definition

A set $I \subseteq (0,1] \times [0,\infty)$ is called north-west if $(a,b) \in I$ implies $(0,a] \times [b,\infty) \subseteq I$.



(a) Skewed (b) Square (c) Disjunction of squares (d) Curved set

We associate to each north-west set its frontier

$$f_I(a) \;:=\; \inf\{b \geq a \mid (a,b) \in I\}.$$

Reverse engineering II

Fix a north-west set I. The north-west adjuster

$$G_I(a,b) \;:=\; \frac{\mathbf{1}_{\{(a,b)\in I\}}}{1-\exp\left(\int_0^1 \frac{1}{x-f_I(x)}\,\mathrm{d}x\right)} \;=\; \frac{\mathbf{1}_{\{b\geq f_I(a)\}}}{1-\exp\left(\int_0^1 \frac{1}{x-f_I(x)}\,\mathrm{d}x\right)}.$$

is witnessed by the north-west strategy S_I, which takes position

$$S_I(\omega_0,\dots,\omega_{t-1}) \;=\; \frac{\frac{1}{f_I(m)-m}\exp\left(\int_0^m \frac{1}{a-f_I(a)}\,\mathrm{d}a\right)}{1-\exp\left(\int_0^1 \frac{1}{x-f_I(x)}\,\mathrm{d}x\right)} \qquad \text{where } m=\min_{0\leq s<t}\omega_t$$

until $\omega_t \geq f_I(m)$.

Buys more shares when the global minimum sinks.

What just happened

Buy low, sell high:

- The intuitive extension fails
  - Mixtures of threshold guarantees are strictly dominated.
  - We need temporal reasoning to appreciate that
- We classified candidate guarantees using a simple formula
  - (≤ 1) Attainable adjuster
  - ($= 1$) Admissible adjuster
  - (> 1) Not an adjuster
- The formula is not explicitly temporal
- We reverse engineered a strategy for each guarantee
  - Mixture of north-west-set strategies

Open problems

- Sell high, buy low, then sell high again.
- ...

Thank you!