Quarto!

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TCS Module 2: Software Systems
Tuesday 20th January, 2026

About

About Me

- Professor of mathematical machine learning at UT
- Senior researcher in machine learning group at CWI
- BSc CS & AI. MSc Logic. PhD computer science

About Quarto

- Quarto published in 1991
- Many prizes since (and before!)
- Solved in 1998 by Luc Goossens

I solved Quarto in 2010 as a hobby project

Goals of this talk

- Fun and empowering toolbox:
  - Combinatorial game theory
  - Academic programming
- Nice example of brain vs computational power:
  - Thought-assisted combinatorial search
  - Combinatorial-search-assisted thought
- Fascinating symmetries

Two Styles

Roughly two approaches

1. Search
   - backward induction
   - alpha-beta pruning
   - negascout
   - ...
2. AI/RL
   - deep neural networks
   - self-play
   - policy gradient
   - MCTS
   - ...

Outline

1. Quarto crash course
2. The value of Quarto
3. Playing Optimally
4. Conclusion

Rules: the pieces and Quarto

- The **pieces** are the 16 realisations of four binary properties:

$$\underbrace{\{\text{dark}, \text{light}\}}_{\text{colour}} \times \underbrace{\{\text{tall}, \text{short}\}}_{\text{height}} \times \underbrace{\{\text{round}, \text{square}\}}_{\text{shape}} \times \underbrace{\{\text{hollow}, \text{solid}\}}_{\text{consistency}}$$

- Four pieces form **Quarto** if they agree on a property.

$$Q\{p, q, r, s\} \qquad \text{iff} \qquad p_i = q_i = r_i = s_i \quad \text{for some property } i$$

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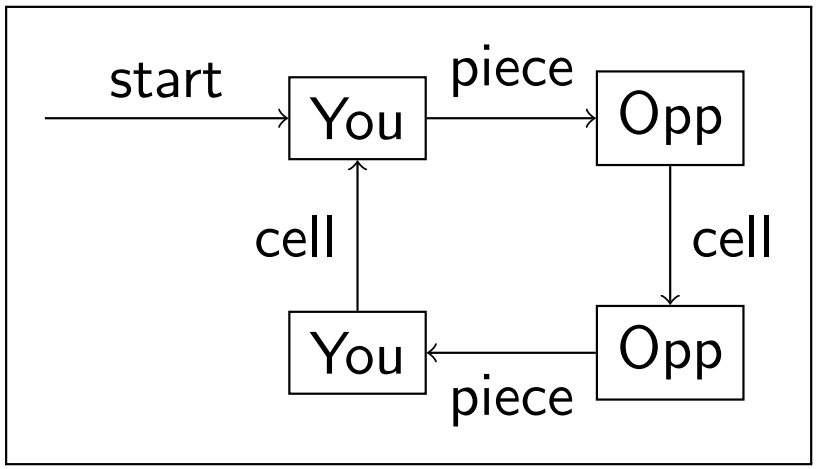
Rules: board, turns and winning

- The **board** has 4×4 cells. Initially empty. Pieces are put aside.
- The game proceeds in **rounds**. Each round has two **plies**:
  - One player gives an unused piece to the other player.
  - The other player places that piece on an empty cell.



- **Win** by forming Quarto in a row, column or (co)diagonal.
- **Draw** when all pieces placed without Quarto.

Questions

- What is the *value of the game*? (i.e. when both players play optimally, does the starting player win, lose or draw?)
- How to play the optimal strategy?

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Naive approach

$$\max_{p_1} \min_{c_1} \min_{p_2} \max_{c_2} \max_{p_3} \ldots \min_{p_{16}} \max_{c_{16}} V(p_1 c_1 \cdots p_{16} c_{16})$$

where

$$V(p_1 c_1 \cdots p_{16} c_{16}) = \begin{cases} -\infty & \text{You disobeyed the rules} \\ -1 & \text{You lose} \\ 0 & \text{Game is a draw} \\ +1 & \text{You win} \\ +\infty & \text{Opp disobeyed the rules} \end{cases}$$

Only $16^{32} \approx 3.4 \cdot 10^{38}$ operations.

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Only $(16!)^2 \approx 4.4 \cdot 10^{26}$ when enforcing the rules.

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Only $(16!)^2 \approx 4.4 \cdot 10^{26}$ when enforcing the rules.
Way too many. Q: Any ideas?

Exploiting positionality (game DAG instead of game tree)

In Quarto, the moves from and payoffs in any state depend only on the current position, and not on how the players got there.

```
 1: function VAL(b)
 2:     if ISQ(b) return WIN
 3:     if ISFULL(b) return DRAW
 4:     if we stored that b has value v then return v
 5:     if b has given piece p then
 6:         v ← max_{c∈cells(b)} VAL(b[p@c])
 7:     else
 8:         v ← max_{p∈pieces(b)} −VAL(b ⊕ p)
 9:     end if
10:     store that b has value v
11:     return v
12: end function
```

We now need $9.9 \cdot 10^{16}$ operations. Still no cigar.

Exploiting symmetries

Some positions are *equivalent*. It suffices to evaluate only one member of each equivalence class.

- Piece symmetries
- Board symmetries

Piece symmetries

Definition (Piece Symmetry)

A *piece symmetry* is a mapping of the 16 pieces to the 16 pieces that preserves Quarto's.

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Q: Find piece symmetries

Fact

There are $4!\,2^4 = 384$ *piece symmetries.*

- *the* 4 *properties can be reordered arbitrarily*
- *the* 2 *values of each property can be flipped*

Board symmetries

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A *board symmetry* is a mapping of the 16 board cells to the 16 board cells that preserves Quarto's.

A board symmetry must map rows/columns to rows/columns and (co)diagonals to (co)diagonals.

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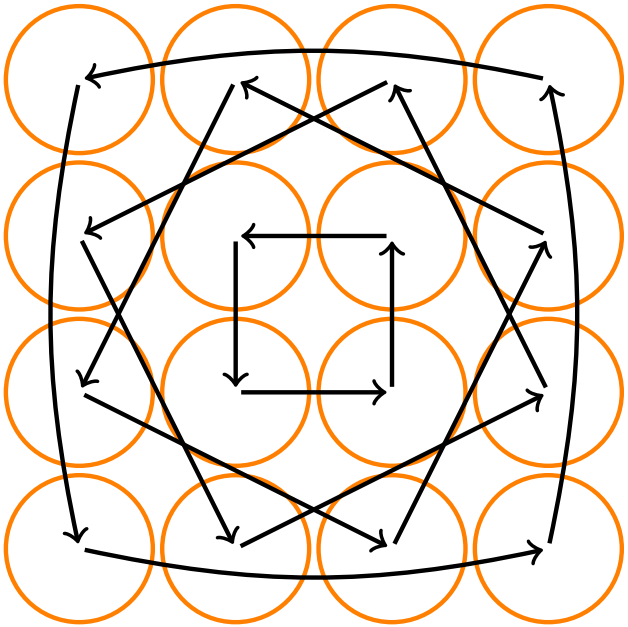
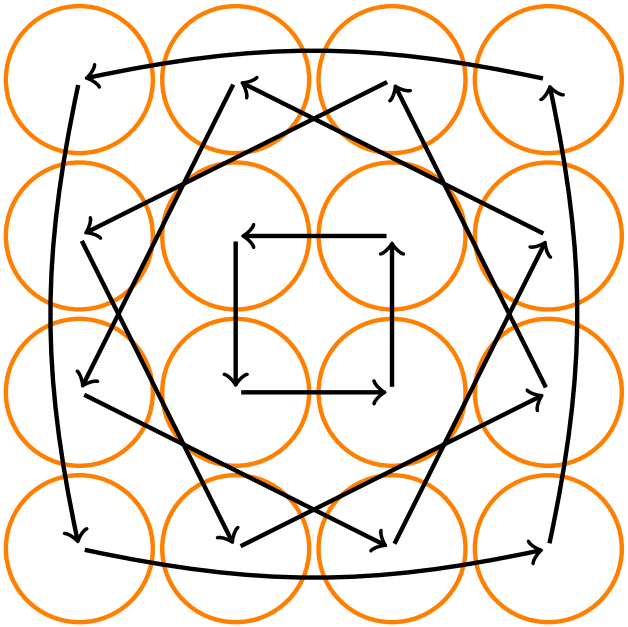
Q: Find board symmetries

Finding board symmetries



counter clockwise rotation

Finding board symmetries



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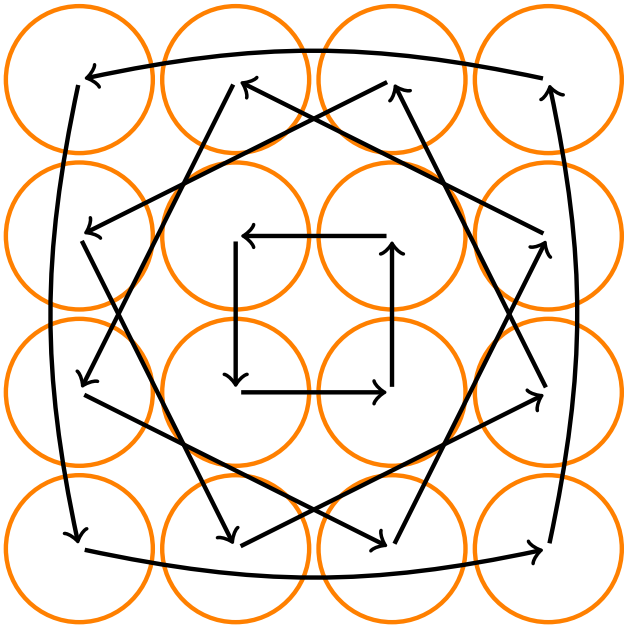

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Q: What about clockwise rotation? A: Rotate ccw thrice

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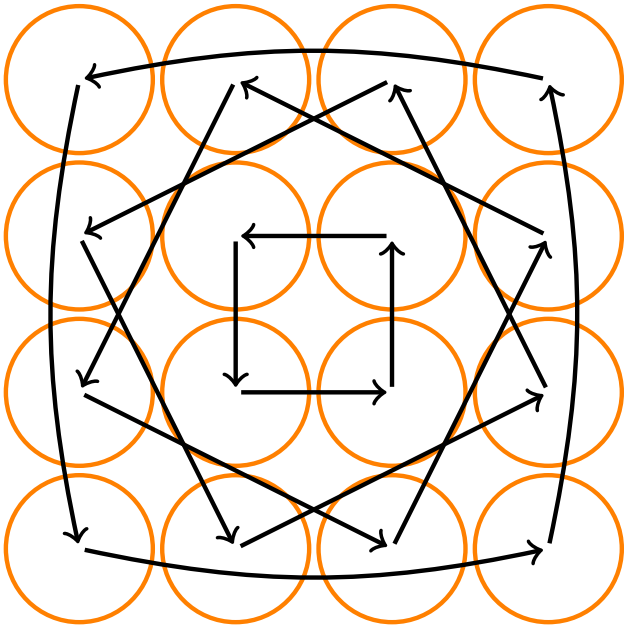

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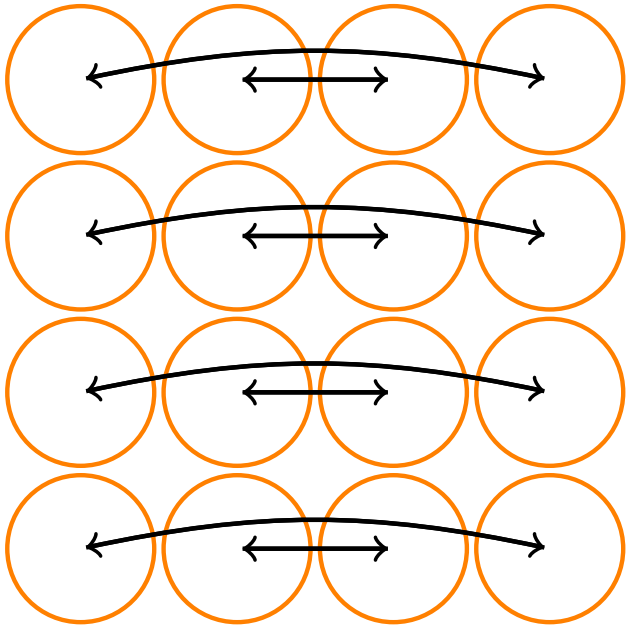

mirror over vertical axis

Q: What about clockwise rotation? A: Rotate ccw thrice

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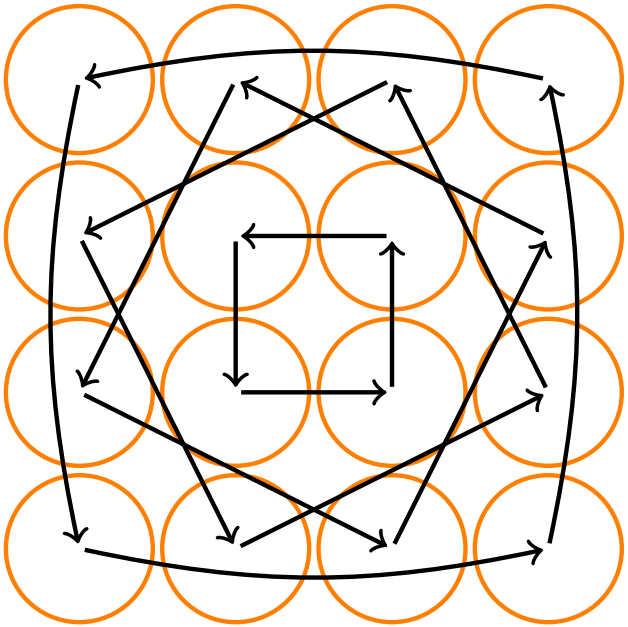

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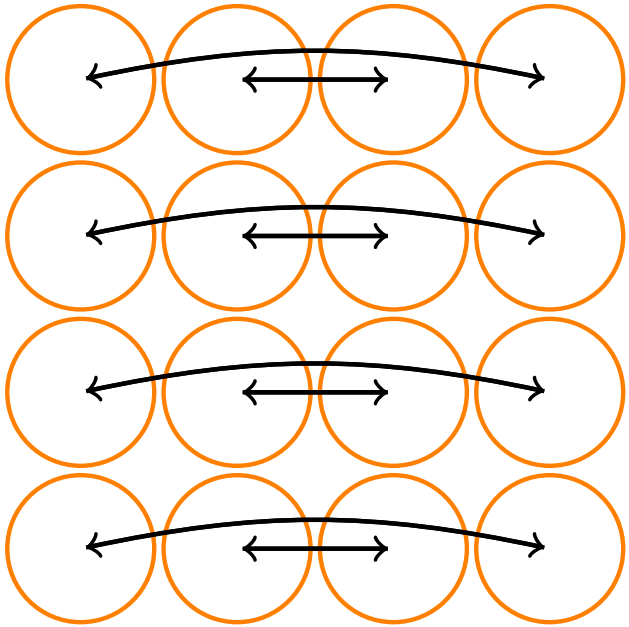

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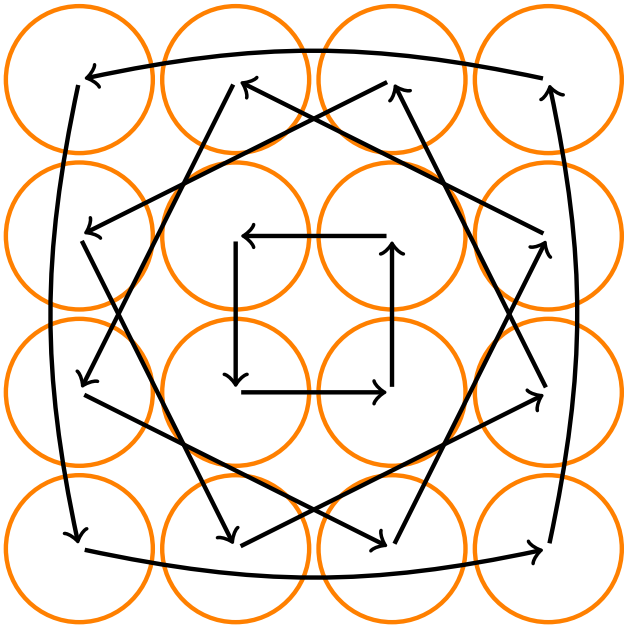

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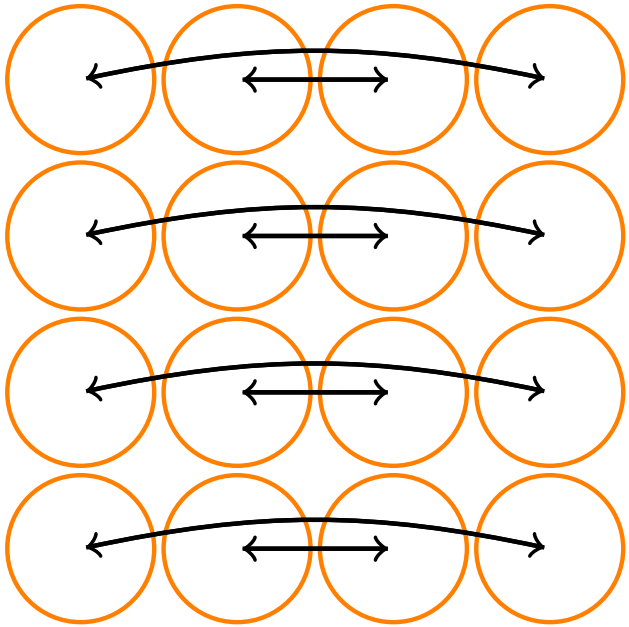

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Q: Mirror over diagonal? A: rotate cw, then mirror

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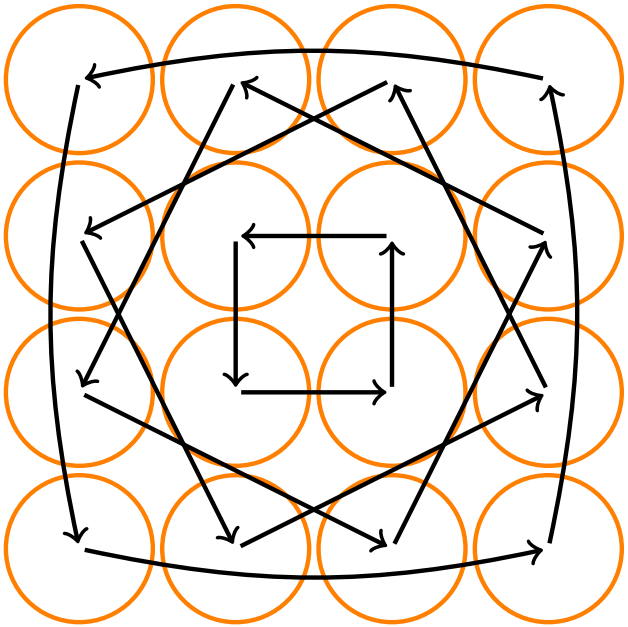

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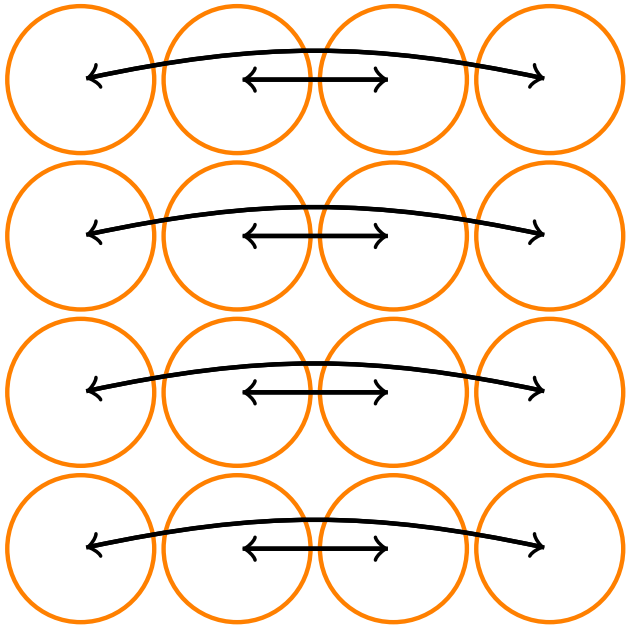

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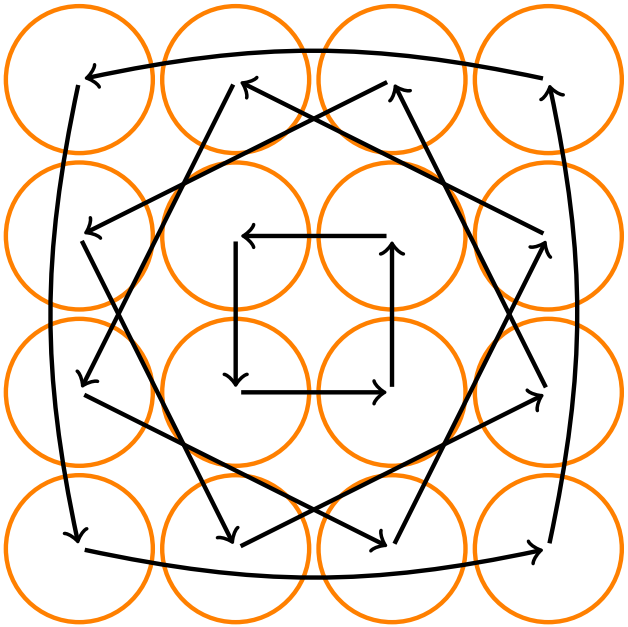

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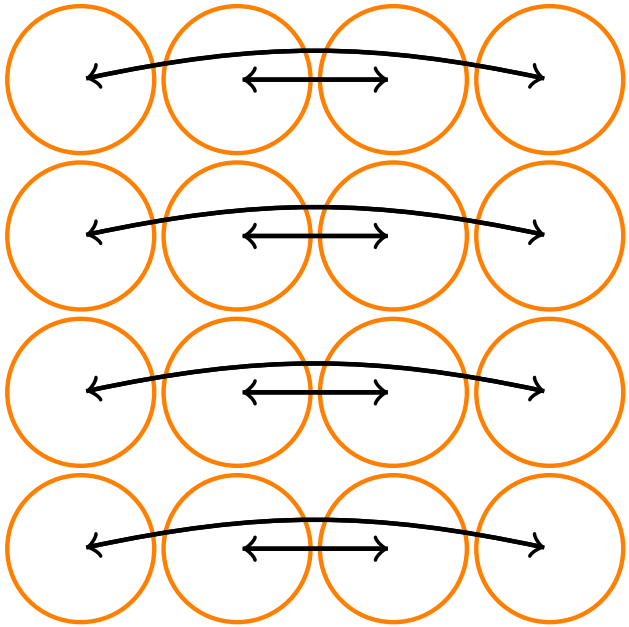

mirror over vertical axis

Q: What about clockwise rotation? A: Rotate ccw thrice
Q: Mirror over diagonal? A: rotate cw, then mirror
Q: Are there other board symmetries?
Q: How to even approach such a question?

Exhaustive enumeration

```
 1: procedure ENUM_SYM(M)
 2:     if M violates group structure then return
 3:     if |M| = 16 then
 4:         print M
 5:     else
 6:         choose a free source cell i
 7:         for each free target cell j do
 8:             ENUM_SYM(M[i → j])
 9:         end for
10:     end if
11: end procedure
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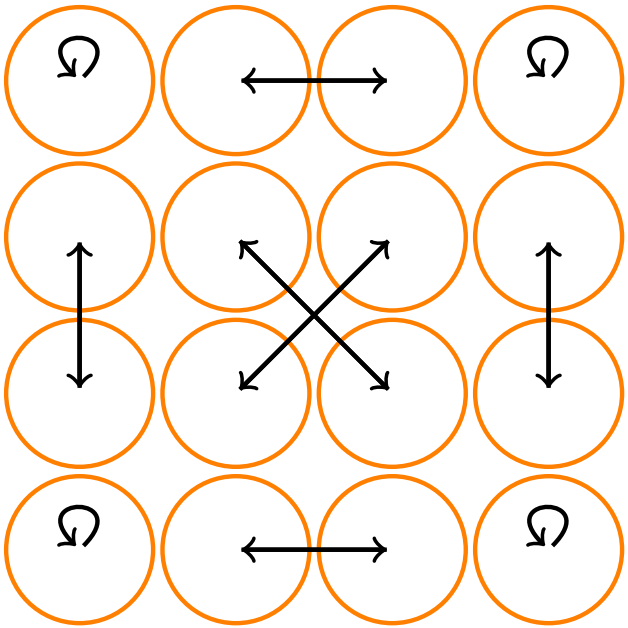
Fact

There are 32 board symmetries.

Finding board symmetries (ctd)



mid flip

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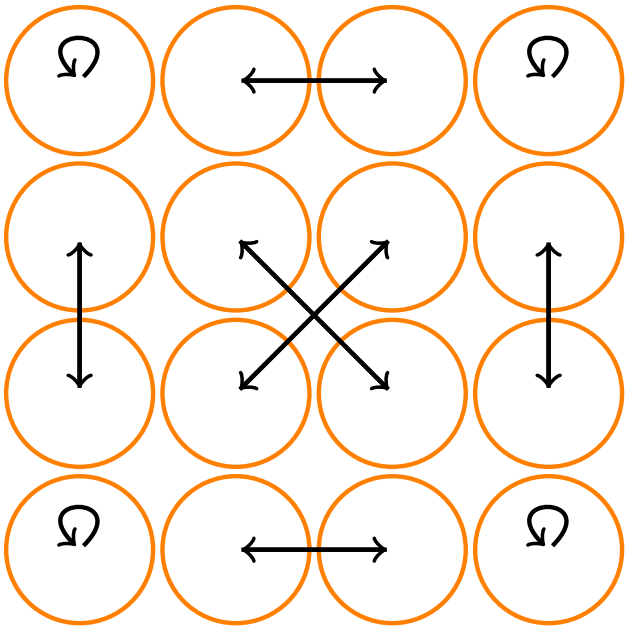

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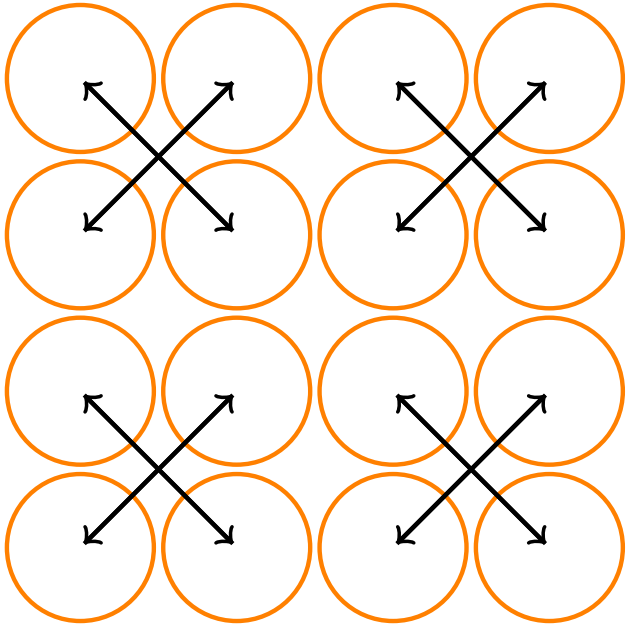
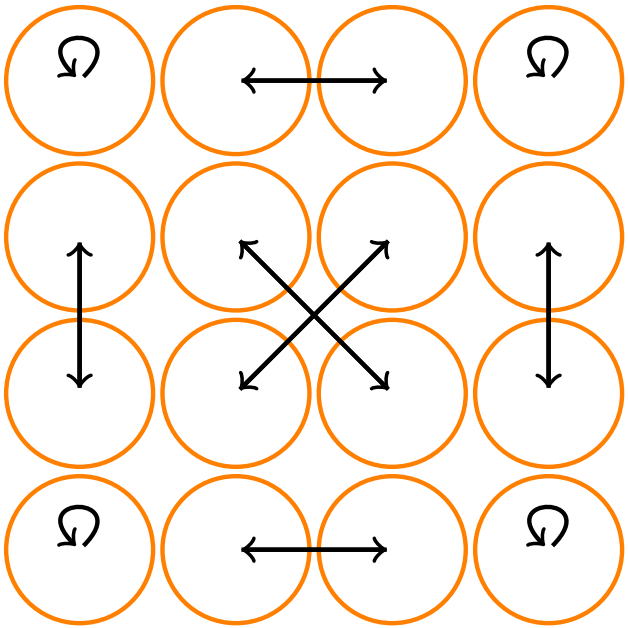


inside out

Finding board symmetries (ctd)



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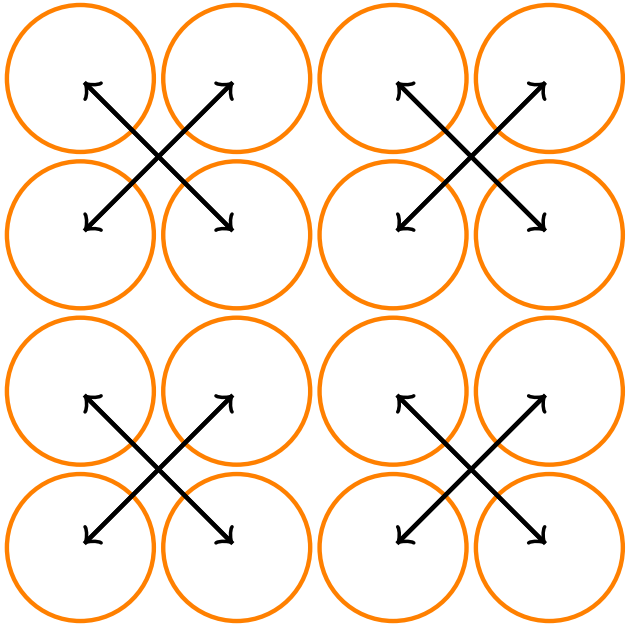
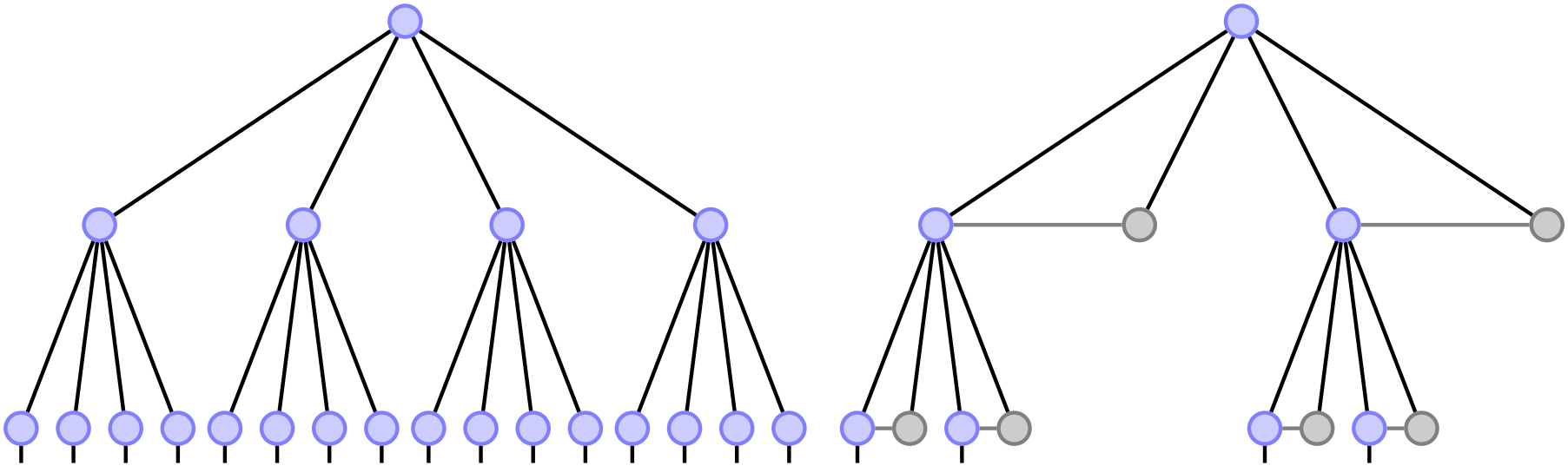


inside out

[Goo04] found 16 (inside out), and [Bro05] found 16(mid flip).

Symmetry benefits

Identifying d positions may *divide* the degree by d. Exponential gain.



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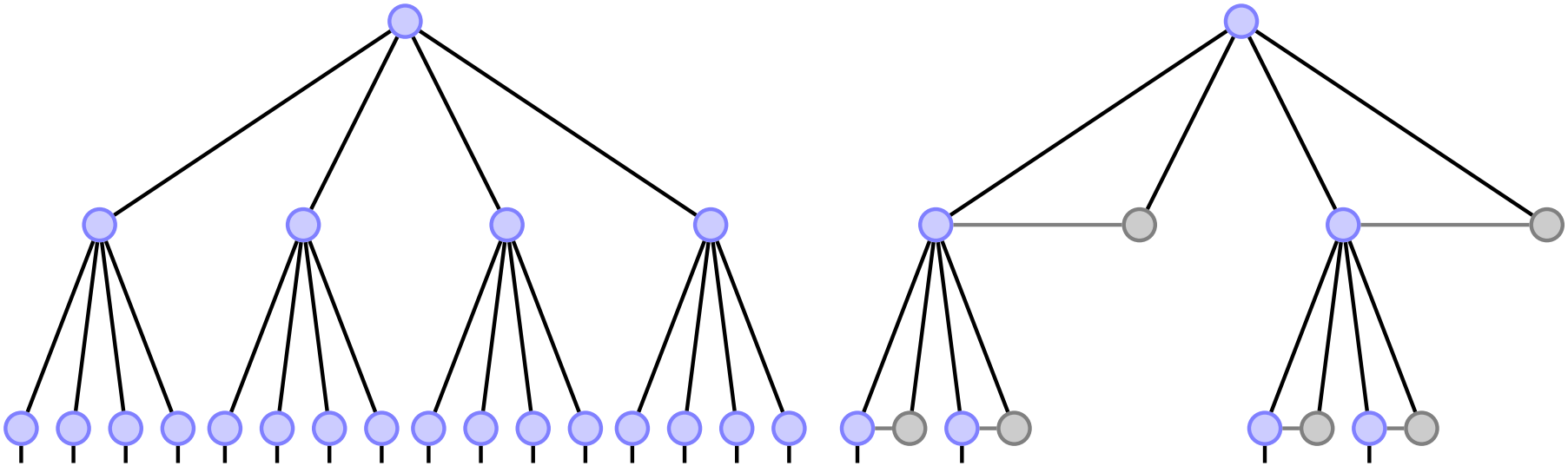
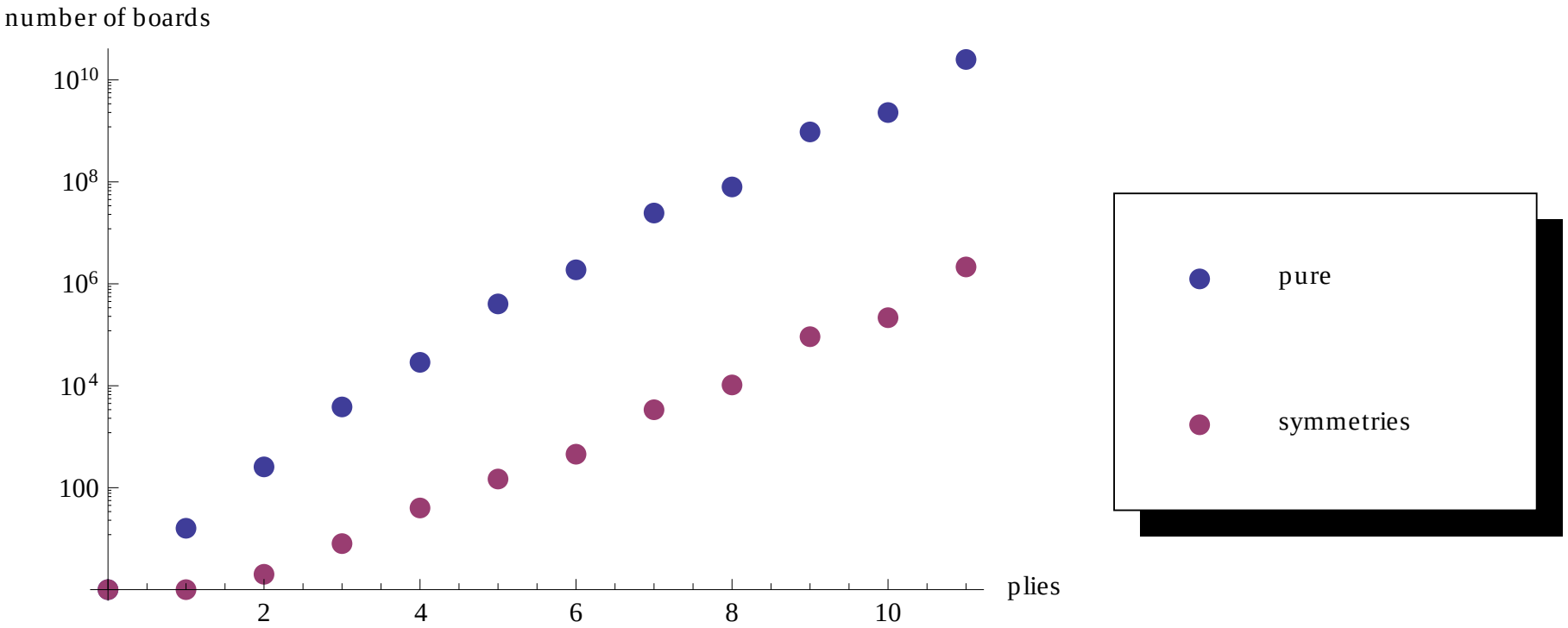


Q: Will it always? If not, what is the least-favourable reduction?

Symmetry benefits ctd.



Good, but we need to explore 32 plies!

How to exploit symmetries

CANONISE picks a canonical representative of each equivalence class.

```
 1: function VAL(b)
 2:     if ISQ(b) return WIN
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 4:     b ← CANONISE(b)
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The final trick

Alpha-beta pruning. Rule of thumb: explores only the square root of the original number of positions.

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Fact

The value of Quarto is draw.

Software finds out in 147 minutes on this laptop.

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Implementing the optimal strategy

- So far, we computed the value of the empty board.
- But to play, we need to evaluate any board.
- We can evaluate positions at ≥ 10 plies from scratch in < 5 seconds.
- There are only 106 156 distinct positions at < 10 plies.
- Compute, once and for all, the value of *all of them*.
- Took about 2 weeks on this laptop.
- Now we can evaluate any position fast.
- To play, choose the child that evaluates to the value of the parent.

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Questions

- Once you have the optimal strategy in tabular form, can one extract an *interpretable* rule? How do we know there even is one?
- What really is the goal of your project? Beat your opponents?

Questions?

Kevin S. Brown.
414298141056 quarto draws suffice!
http://www.mathpages.com/home/kmath352.htm, June 2005.

Luc Goossens.
Quarto.
http://web.archive.org/web/20041012023358/http://ssel.vub.ac.be/Members/LucGoossens/quarto/quartotext.htm, October 2004.