Game AI & Search

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CWI & UT

Credentials

- Quarto
- Quoridor
- Scrabble
- some research papers on learning

Why are we here today?

Games

- Games as objects for studying learning about
- Game theory/methods as engines driving learning methods


THERE IS A GAME ENGINE
INSIDE YOUR GAME ENGINE
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PAC Learning

Definition of Learning

Suitable for passive and interactive learning

Definition of Learning [Valiant'84]

- Start with any computational problem. (However simple)
  - Sorting
  - Shortest path in graph, planning in MDP
  - Least squares, curve fitting
  - Saddle point (of matrix game)
  - Backward induction (of game DAG)
  - Linear / convex / submodular optimisation
  - ...
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- Questions: statistics (# samples), computation (algorithm design)

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- Approximation error: ϵ
- Confidence: δ
- Sample complexity: m
- Computational complexity: $O(n)$

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So we can reason about optimality in handling:

- Features of original computational problem
- Shape constraints assumptions (between means)
- (Non-)parametric assumptions on sampling processes
- Constraints on experiments
- Strategies/templates for dealing with uncertainty, “inference”

Why Games, Then?

Games provide a series of natural problems

- Beyond convex hull of what we know how to do $\Rightarrow$ Challenging
- Yet with hint of mathematical tractability $\Rightarrow$ Ripe
- Applications plentiful $\Rightarrow$ Useful

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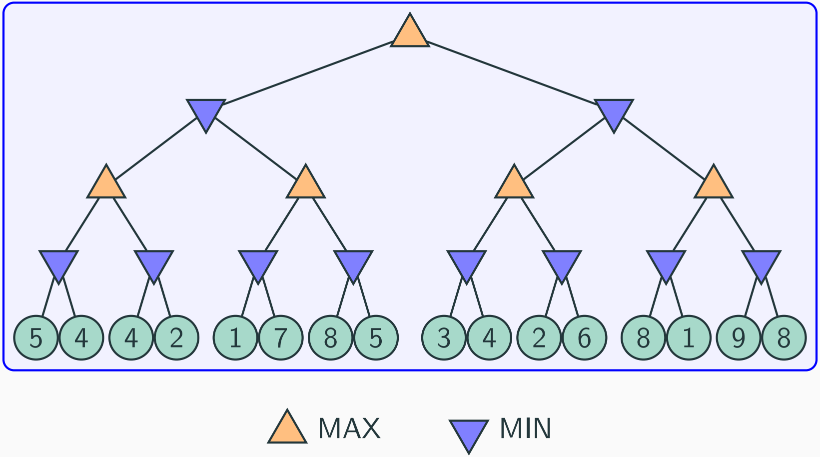
- Learning and Games: cross fertilization
- Learning results inspire systems
- Ambition to explain success of current game systems

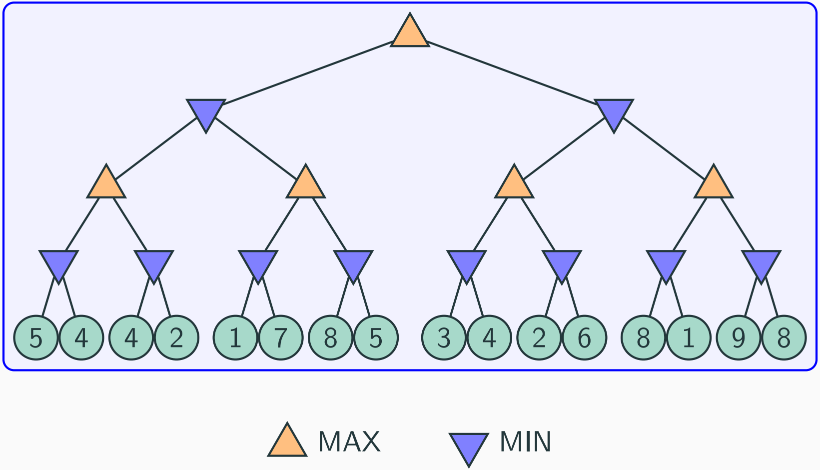
Tree Search

Deterministic Problem



Problem
Given leaf values $\boldsymbol{\mu}$, compute $i^*(\boldsymbol{\mu}) := \operatorname*{argmax}_{a_1} \min_{a_2} \max_{a_3} \min_{a_4} \mu_{a_1 a_2 a_3 a_4}$

Backward Induction Computation



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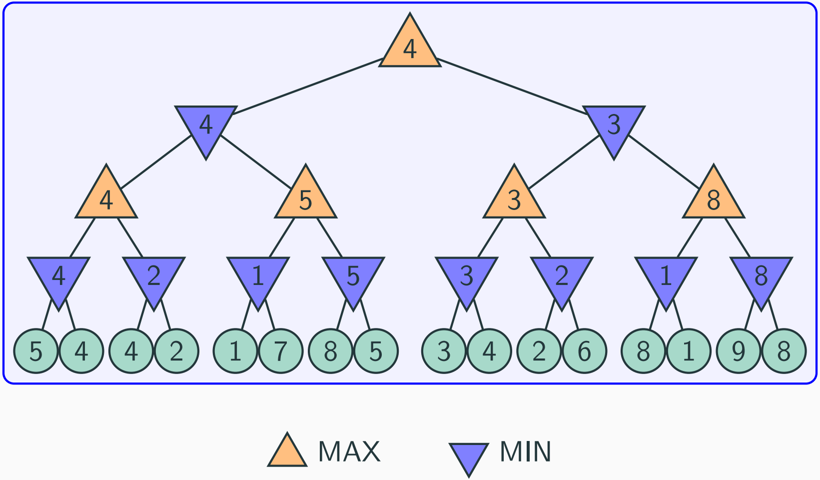
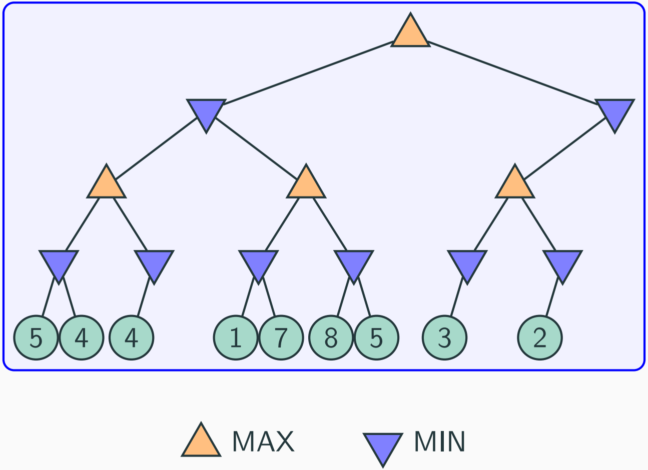
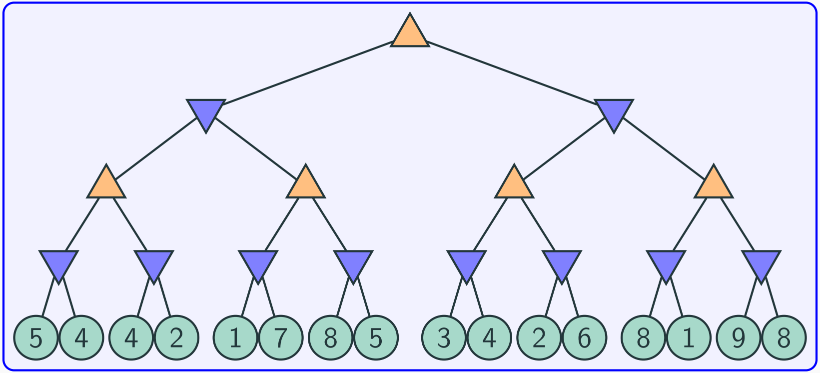


Alpha-beta Pruning



Model (Teraoka, Hatano, and Takimoto, 2014)



Maximin Action Identification Problem
Find best move at root from samples of leaves.

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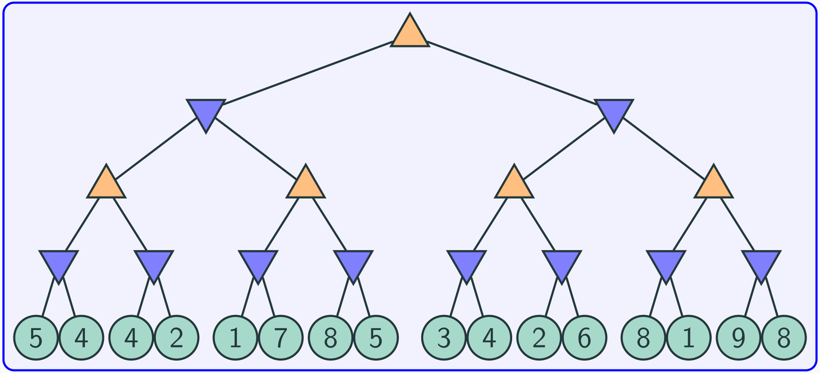

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Protocol

For $t = 1, 2, \ldots, \tau$:

- Learner picks a leaf $L_t \in \mathcal{L}$.
- Learner sees $X_t \sim \mu_{L_t}$

Learner recommends action $\hat{I}$

Objectives

Learner is δ-PAC if

$$\forall \boldsymbol{\mu} : \mathop{\mathbb{P}}_{\boldsymbol{\mu}}\left(\tau < \infty \wedge \hat{I} \neq i^*(\boldsymbol{\mu})\right) \leq \delta$$

Goal: minimise sample complexity $\mathbb{E}_{\boldsymbol{\mu}}[\tau]$ over all δ-PAC strategies.

Main Theorem I: Lower Bound

Define the *alternatives* to $\boldsymbol{\mu}$ by $\mathrm{Alt}(\boldsymbol{\mu}) = \{\boldsymbol{\lambda} | i^*(\boldsymbol{\lambda}) \neq i^*(\boldsymbol{\mu})\}$.

NB here i^* is best action at the root

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Theorem (Castro 2014; Garivier and Kaufmann 2016)

Fix a δ-correct strategy. Then for every bandit model $\boldsymbol{\mu}$

$$\mathbb{E}_{\boldsymbol{\mu}}[\tau] \;\geq\; T^*(\boldsymbol{\mu}) \ln \frac{1}{\delta}$$

where the characteristic time $T^*(\boldsymbol{\mu})$ is given by

$$\frac{1}{T^*(\boldsymbol{\mu})} \;=\; \max_{\boldsymbol{w} \in \triangle_K} \min_{\boldsymbol{\lambda} \in \text{Alt}(\boldsymbol{\mu})} \sum_{i=1}^{K} w_i \, \text{KL}(\mu_i \| \lambda_i).$$

Main Theorem II: Algorithm

Idea is consider the oracle weight map

$$\boldsymbol{w}^*(\boldsymbol{\mu}) \;:=\; \operatorname*{argmax}_{\boldsymbol{w}\in\triangle_K} \min_{\boldsymbol{\lambda}\in\mathrm{Alt}(\boldsymbol{\mu})} \sum_{i=1}^{K} w_i \,\mathrm{KL}(\mu_i\|\lambda_i)$$

and track the plug-in estimate: sample leaf $L_t \sim \boldsymbol{w}^*(\hat{\boldsymbol{\mu}}(t-1))$.

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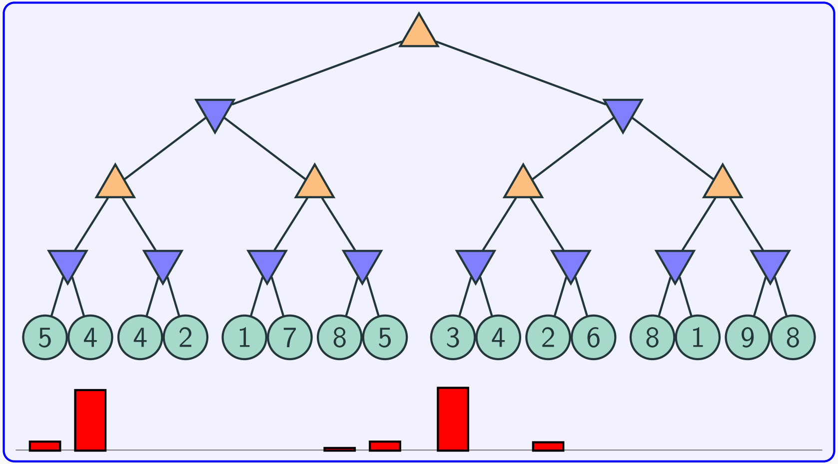
Theorem (Degenne and Koolen, 2019)

Take set-valued interpretation of argmax *defining* $\boldsymbol{w}^*$. *Then* $\boldsymbol{\mu} \mapsto \boldsymbol{w}^*(\boldsymbol{\mu})$ *is upper-hemicontinuous and convex-valued. Suitable tracking ensures that as* $\hat{\boldsymbol{\mu}}(t) \to \boldsymbol{\mu}$, *any choice* $\boldsymbol{w}_t \in \boldsymbol{w}^*(\hat{\boldsymbol{\mu}}(t-1))$ *have*

$$\min_{\boldsymbol{w}\in\boldsymbol{w}^*(\boldsymbol{\mu})} \|\boldsymbol{w}_t - \boldsymbol{w}\|_\infty \to 0$$

Track-and-Stop is asymptotically optimal: $\limsup_{\delta\to 0} \frac{\mathbb{E}_{\boldsymbol{\mu}}[\tau]}{\ln\frac{1}{\delta}} = T^*(\boldsymbol{\mu})$.

Example with oracle weights



On Computation

To compute a gradient (in $\boldsymbol{w}$) we need to differentiate

$$\boldsymbol{w} \;\mapsto\; \min_{\boldsymbol{\lambda} \in \mathrm{Alt}(\boldsymbol{\mu})} \sum_{i=1}^{K} w_i \, \mathrm{KL}(\mu_i \| \lambda_i)$$

An optimal $\boldsymbol{\lambda} \in \mathrm{Alt}(\boldsymbol{\mu})$ can be found by binary search for common value plus tree reasoning in $O(|\mathcal{L}|)$.

AI

But where do we put the AI?

One possible answer:

End-to-End-Deep-Learning
Excellent performance at relatively little understanding. (Silver et al., 2017)

Another possible answer:

Algorithms with Predictions
Heuristics and guarantees can typically be combined.

- Algorithm with worst-case performance guarantee . . .
- . . . that improves whenever predictions are good
- Example: node ordering for (α, β) pruning.

Open Problems

Problem: Multiple answers and $\epsilon > 0$

We have seen Best Action Identification. Instance of PAC learning with $\epsilon = 0$.

Lower bound complexity essentially governed by certain gaps between leaf means.

We can learn **faster** if we accept any ϵ-optimal move.

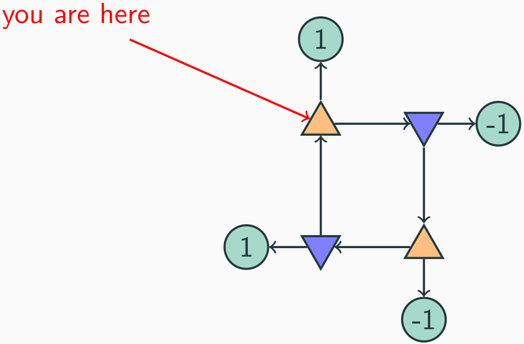
Theory currently underdeveloped:

- Asymptotic optimality as $\delta \to 0$ results exist (Degenne and Koolen, 2019) but are of questionable *practicality*.
- My perspective: lower bounds need sharpening
- Practical schemes based on confidence intervals Kaufmann and Koolen, 2017

Problem
What is the sample complexity of ϵ Best Action Identification?

Problem: Cyclic game graphs

Consider a two-player zero-sum **cyclic** game graph.



Convention: cycling forever results in value 0.

Problem
What is the sample complexity of finding the best move in a cyclic game?

Problem: Facing the Curse of Dimensionality

Game trees are intractably big. Practical systems (Silver et al., 2017)

- Progressively expand strategic horizon
- Cluster similar nodes
- Exploit background knowledge
-

Problem: Mixtures and $\epsilon > 0$

With Julia Olkovskaya (in progress) we found an intriguing possibility.

Observation
In bandits, it appears much easier to find a ϵ-good mixture of arms than to find a **individual** ϵ best arm.

- Statistics improve
- Computation streamlines

Problem
What is the sample complexity of ϵ- Best Mixture of Actions Identification?

Problem: Oracle Strategy Computation

Algorithms based on lower bound problems of the form

$$\operatorname*{argmax}_{\boldsymbol{w} \in \triangle_K} \min_{\boldsymbol{\lambda} \in \mathrm{Alt}(\boldsymbol{\mu})} \sum_{i=1}^{K} w_i \, \mathrm{KL}(\mu_i \| \lambda_i)$$

With the underlying deterministic problem determining what Alt means.

Problem
As a function of Alt, *what is the computational complexity of optimisation of the above objective?*

Gradient descent vs cutting plane vs interior point methods? Can we do Nesterov acceleration? Scaling with the dimension K and perhaps the problem rank (Kaufmann and Koolen, 2021)?

Conclusion

Games and **Learning** fertile research areas with interesting, challenging, promising intersection.

References

Castro, R. M. (Nov. 2014). “Adaptive sensing performance lower bounds for sparse signal detection and support estimation”. In: *Bernoulli* 20.4, pp. 2217–2246.

Degenne, R. and W. M. Koolen (Dec. 2019). “Pure Exploration with Multiple Correct Answers”. In: *Advances in Neural Information Processing Systems (NeurIPS) 32*. Ed. by H. Wallach, H. Larochelle, A. Beygelzimer, F. d’Alché-Buc, E. Fox, and R. Garnett. Curran Associates, Inc., pp. 14591–14600.

Garivier, A. and E. Kaufmann (2016). “Optimal Best arm Identification with Fixed Confidence”. In: *Proceedings of the 29th Conference On Learning Theory (COLT)*.

Kaufmann, E. and W. M. Koolen (Nov. 2021). “Mixture Martingales Revisited with Applications to Sequential Tests and Confidence Intervals”. In: *Journal of Machine Learning Research* 22.246, pp. 1–44.

Kaufmann, E. and W. M. Koolen (Dec. 2017). "Monte-Carlo Tree Search by Best Arm Identification". In: *Advances in Neural Information Processing Systems (NeurIPS) 30*. Ed. by I. Guyon, U. V. Luxburg, S. Bengio, H. Wallach, R. Fergus, S. Vishwanathan, and R. Garnett, pp. 4904–4913.

Silver, D. et al. (Oct. 2017). "Mastering the game of Go without human knowledge". In: *Nature* 550, pp. 354–359.

Teraoka, K., K. Hatano, and E. Takimoto (2014). "Efficient Sampling Method for Monte Carlo Tree Search Problem". In: *IEICE Transactions* 97-D.3, pp. 392–398.